A Ferrand-Obata theorem for rank one parabolic geometries

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Abstract. The aim of this article is the proof of the following result:

Theorem. *Let M be a connected manifold endowed with a regular Cartan geometry (M, B, ω) modelled on the boundary $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{K}}$ of a d-dimensional hyperbolic space $\mathbf{H}^d_{\mathbb{K}}$ over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or the algebra of octonions $\mathbb{O}$). If the group of automorphisms $Aut(M, \omega)$ does not act properly on M, then M is geometrically isomorphic to:*
- $\mathbf{X}$ *if M is compact.*
- $\mathbf{X}$ *minus a point in the other cases.*

1 Introduction

At the very begining of the seventies, several works inspired by the so called *Lichnerowicz's conjecture* about the conformal group of Riemannian manifolds, led to the following result:

Theorem 1 (Ferrand-Obata). *[Fe1],[Fe2],[Ob]*

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. If the group of conformal transformations of (M, g) does not act properly on M, then (M, g) is conformally equivalent to:
- *the standard conformal sphere if M is compact.*
- *the Euclidean space if M is noncompact.*

Let us recall here that the action of a group G, by homeomorphisms on a manifold M, is said to be *proper* if for every compact subset $K \subset M$, the set:

$$G_K = \{g \in G \mid g(K) \cap K \neq \emptyset\}$$

has compact closure in $Homeo(M)$ (where $Homeo(M)$, the group of homeomorphisms of M is endowed with the compact-open topology). We do not suppose *a priori* that G is closed in $Homeo(M)$.

Let us precise that the correct and definitive proof in the noncompact case was achieved by J.Ferrand alone in [Fe2]. The proof of M.Obata dealt only with the compact case, under the more restrictive asumption that the action of the *identity component* of the conformal group, is nonproper. A different proof of Theorem 1 has been proposed by R.Schoen in [Sch]. The methods involved allowed moreover to get a similar result for CR structures:

Theorem 2 (Schoen). *Let M^{2n+1}, $n \geq 1$, be a manifold endowed with a strictly pseudo-convex CR structure. Then , if the group of CR automorphisms of M does not act properly on M, then:*

- *M is CR-diffeomorphic to the standard CR sphere if M is compact.*
- *M is CR-diffeomorphic to the Heisenberg group, endowed with its standard structure, if M is noncompact.*

1.1 Cartan geometries

Although the methods used by J.Ferrand and R.Schoen are completely different, the statements of Theorems 1 and 2 are very analogous, and let suspect that they are two aspects of a more general result. From the geometric point of view, there is a concept which unify, in dimension $n \geq 3$, Riemannian conformal structures and strictly pseudo-convex CR structures: that of *Cartan geometry.*

Intuitively, a Cartan geometry is the data of a manifold infinitesimally modelled on some homogeneous space $\mathbf{X} = G/P$, where G is a Lie group and P a closed subgroup of G. More technically, a Cartan geometry on a manifold M, modelled on the homogeneous space $\mathbf{X} = G/P$, is the data of:
(i) a principal P-bundle $B \to M$ over M.
(ii) a 1-form ω on B, with values in the Lie algebra $\mathfrak{g}$, called *Cartan connexion*, and satisfying the following conditions:

- At every point $p \in B$, ω_p is an isomorphism between T_pB and $\mathfrak{g}$.
- If $X^\dagger$ is a vector field of B, comming from the action by right multiplication of some one-parameter subgroup $t \mapsto Exp_G(tX)$ of P, then $\omega(X^\dagger) = X$.
- For every $a \in P$, ${R_a}^*\omega = Ad(a^{-1})\omega$ (R_a standing for the right action of a on B).

A Cartan geometry on a manifold M will be denoted by (M, B, ω).

A lot of classical geometric structures can be interpreted in terms of Cartan geometry. The most famous ones are Riemannian metrics (resp. pseudo-Riemannian metrics of signature (p,q)). In this case, the space $\mathbf{X}$ is just the Euclidean space (resp. the Minkowski space of signature (p,q)), G is the group of isometries $SO(n) \ltimes \mathbb{R}^n$ (resp. $SO(p,q) \ltimes \mathbb{R}^n$), and P the normal subgroup constituted by the translations.

Since the definition of a Cartan geometry is not very intuitive, two natural problems arise at once. The first is the interpretation of the data of a Cartan connection ω on a principal bundle $B \to M$, as an underlying geometric structure on M. In a lot of interesting geometric situations, such an interpretation is available (see for instance [CS], [M], [T2], and references therein). The second interesting problem is to know, if a given underlying structure on a manifold M, determines canonically a Cartan geometry. This problem, known as the *equivalence problem*, is generally quite difficult. It was solved by E.Cartan himself for conformal Riemannian structures, in dimension $n \geq 3$, and for strictly pseudo-convex CR structures ([Ca1] in dimension 3, [T1] and [Ch] for dimensions $n \geq 3$. See also [Ko] and [Sha] for the conformal case). Otherwise stated, if a manifold M of dimension $n \geq 3$, is endowed with a conformal class of Riemannian metrics (resp. with a strictly pseudo-convex CR structure), one is able to build a P-principal bundle B over M, and a Cartan connection ω on it. In the conformal (resp. CR) case, the model space $\mathbf{X}$ is the conformal sphere $\mathbf{S}^n = \partial \mathbf{H}_{\mathbb{R}}^{n+1}$ (resp. the CR sphere $\mathbf{S}^{2n+1} = \partial \mathbf{H}_{\mathbb{C}}^{n+1}$), the group G is the Moebius group $SO(1, n+1)$ (resp. the group $SU(1, n+1)$), and P is a parabolic subgroup: the stabilizer of a point on $\mathbf{X}$. Moreover, if one requires that it satisfies suitable normalizations conditions, the Cartan connection ω is defined uniquely. Thus, any conformal diffeomorphism (resp. CR diffeomorphism) acts on B preserving the connection ω.

For any Cartan geometry (M, B, ω), we define $Aut(B, \omega)$ as the set of C^1-diffeomorphisms ϕ of B, such that $\phi^* \omega = \omega$. Every ϕ of $Aut(B, \omega)$ commutes with the right action of P on B, so that ϕ induces a diffeomorphism $\overline{\phi}$ of M. The subset of diffeomorphisms of M obtained in this way is denoted by $Aut(M, \omega)$.

There is also a notion of geometric equivalence for two Cartan geometries (M, B, ω) and (N, B', ω'), modelled on the same space $\mathbf{X} = G/P$. We say that M and N are *geometrically isomorphic*, if there is a diffeomorphism $\phi : B \to B'$ such that $\phi^* \omega' = \omega$.

1.2 Statement of results

The aim of the article is the generalization of Theorems 1 and 2 to any Cartan geometry modelled on spaces $\mathbf{X} = G/P$, where G is a simple Lie group of real rank one, with finite center, and P is a parabolic subgroup of G. These spaces X are the boudaries of the different hyperbolic spaces $\mathbf{H}_{\mathbb{K}}^d$, $\mathbb{K}$ standing for the field $\mathbb{R}$ of real numbers, $\mathbb{C}$ of complex numbers, $\mathbb{H}$ of quaternions, or the octonions $\mathbb{O}$. We will make the asumption $d \geq 2$ if $\mathbb{K} = \mathbb{R}$, and $d \geq 1$ otherwise (except for the octonionic case, where the dimension d

is necessarily 2). Implicitely, when we will speak about a Cartan geometry modelled on $\partial \mathbf{H}^d_{\mathbb{K}}$, we will always see $\partial \mathbf{H}^d_{\mathbb{K}}$ as the homogeneous space G/P, where $G = Iso(\mathbf{H}^d_{\mathbb{K}})$, and P is the stabilizer, in $Iso(\mathbf{H}^d_{\mathbb{K}})$, of a point of $\partial \mathbf{H}^d_{\mathbb{K}}$.

Let us recall the groups G involved:

- $G = SO(1,n)$, $d \geq 2$ for $\mathbb{K} = \mathbb{R}$. The space $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{R}}$ is a sphere $\mathbf{S}^{d-1}$.
- $G = SU(1,d)$, $d \geq 1$ for $\mathbb{K} = \mathbb{C}$. The space $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{C}}$ is a sphere $\mathbf{S}^{2d-1}$.
- $G = Sp(1,d)$, $d \geq 1$ for $\mathbb{K} = \mathbb{H}$. The space $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{H}}$ is a sphere $\mathbf{S}^{4d-1}$.
- $G = F_4^{-20}$ if $\mathbb{K} = \mathbb{O}$. The space $\mathbf{X} = \partial \mathbf{H}^2_{\mathbb{O}}$ is a sphere $\mathbf{S}^{15}$.

We can now state our main result:

Theorem 3. *Let (M, B, ω) be a Cartan geometry modelled on $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{K}}$, the boundary of the d-dimensional hyperbolic space over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. We suppose that M is connected, and that the connexion ω is regular. Then if $Aut(M,\omega)$ does not act properly on M, then M is geometrically isomorphic to:*

- $\mathbf{X}$ *if M is compact.*
- $\mathbf{X}$ *minus a point otherwise.*

The hypothesis of regularity is a technical hypothesis on the curvature of the connection ω (which is mild, since satisfied in most of interesting cases). The notion of regularity will be explained in section 3.

The conclusions of Theorem 3 will hold for any geometric structure on a manifold M, for which the equivalence problem has been solved, and from which one is able to define a *regular* canonical Cartan geometry, modelled one one of the spaces $\partial \mathbf{H}^d_{\mathbb{K}}$. Since the geometries involved in Theorem 3 are parabolic geometries, we can use the works done on the equivalence problem for these geometries. We get that the theorem will apply to conformal Riemannian structures, and strictly pseudo-convex CR-structure in dimension $n \geq 3$ (this gives a unified proof for Theorems 1 and 2), but also to partially integrable almost CR-structures ([CS], [M], [T2]), as well as to quaternionic and octonionic contact structures introduced by O.Biquard ([Bi], see also [C]).

1.3 Ideas of the proof and organisation of the paper

The proof of Theorem 3 is based essentially on the understanding of the dynamics of sequences of automorphisms of a manifold M, endowed with a Cartan geometry modelled on some $\partial \mathbf{H}^d_{\mathbb{K}}$. The main point is to prove that if a sequence (f_k) of $Aut(M,\omega)$ does not act properly on M, it has the property:

(P): *there is an open subset $U \subset M$ which collapses to a point under the action of (f_k).*

This is a fundamental property since one can prove that it implies the flatness of the geometry on the open subset U (see Proposition 5).

In fact, J.Ferrand and R.Schoen proved also the property (P) for the sequences of conformal (resp. CR) diffeomorphisms which do not act properly. Let us observe that they both did it using analytical tools (J.Ferrand writes in the introduction of [Fe2] "*In fact,* [Theorem 1] *is not actually concerning the theory of Lie groups and may be considered as a mere theorem of Analysis*").

The way we adopt to prove the property (P) is on the contrary purely geometric. Let us begin, recalling that this property (P) is typical of the sequences of $Iso(\mathbf{H}^d_{\mathbb{K}})$, when they act on the boundary $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{K}}$. That is what is usually called a dynamics of *convergence type* (we say also "North-South" dynamics; see section 2). The main idea of Theorem 3 is to use the Cartan connection to link the dynamical properties of sequences of $Aut(M, \omega)$, to dynamical properties of sequences of G, acting on $\mathbf{X}$. Let us begin with the simple case of a flat Cartan geometry. Such a geometry is more comonly called a $(G, \mathbf{X})$-structure on M. In this case, one can define a developping map $\delta : \tilde{M} \to \mathbf{X}$, which is an immersion, as well as a morphism $\rho : Aut(\tilde{M}, \tilde{\omega}) \to G$ (we refer for example to [Th] for general results on $(G, \mathbf{X})$-structures). Moreover, the equivariance relation $\delta \circ f = \rho(f) \circ \delta$ is satisfied for every $f \in Aut(\tilde{M}, \tilde{\omega})$. This equivariance relation is crucial since it allows to recover, at least locally, the dynamics of a sequence (f_k) of $Aut(\tilde{M}, \tilde{\omega})$ from the dynamics of $\rho(f_k)$ on $\mathbf{X}$ (see, for example, [L] or [FT] as an illustration).

Of course, the Cartan geometry we are looking at is generally not flat, *a priori*. Although all the tools used in the case of $(G, \mathbf{X})$-structures break down in this case, it still remains something of the previous scheme. Let us fix $x_0 \in M$, $\hat{x}_0 \in B$ over x_0, and $o \in \mathbf{X}$. The Cartan connection still allows to define some kind of developping map, denoted $\mathcal{D}^{\hat{x}_0}_{x_0}$. This is a map from the space of curves of M passing through x_0, to the space of curves of $\mathbf{X}$ passing through o. This (classical) procedure will be recalled in section 3.3.

Now, let (f_k) be a sequence of $Aut(M, \omega)$. To simplify and avoid technicalities, we suppose here that (f_k) fixes x_0. In section 5.2, we will explain how to associate to (f_k) some *holonomy sequence* (b_k) of P. The fundamental point is that we still have some equivariance relation:

$$\mathcal{D}^{\hat{x}_0}_{x_0} \circ f_k = b_k \circ \mathcal{D}^{\hat{x}_0}_{x_0} \tag{1}$$

Let us insist on the fact that this is a general principle, which will be

probably useful for the dynamical study of automorphisms of other Cartan geometries than those of this paper. Relation (1) shows a link between the action of (f_k) on the curves of M passing through x_0, and the action of (b_k) on the curves of $\mathbf{X}$ passing through o. Of course, the space of curves of $\mathbf{X}$ (resp. of M) passing through o (resp. through x_0) is to much complicated, and we will restrict ourself to the action on a class of curves, which are distinguished for the geometry under consideration: the geodesics of the Cartan geometry. These geodesics are defined in section 4. They coincide with the conformal geodesics when $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{R}}$, and the chains introduced by E.Cartan when $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{C}}$.

The rough idea to get the property (P), is that there will still be some trace of the "North-South" dynamics, for the action of (b_k) on the set of geodesics passing through o (that is the result of the study done in section 2.5.2). Thanks to the relation (1), this dynamical behaviour will be transmitted to the action of (f_k) on the set of geodesics passing through x_0. We then recover the dynamics of (f_k) around x_0 by some kind of projection.

Let us remark that the "North-South" dynamical behaviour caracterizes the rank one situation. This is basically why Theorem 3 has no analogous for certain higher rank parabolic geometries (see for example [Fr2], which deals with the conformal Lorentzian situation).

As already said, once the property (P) is proved, we get the flatness of the Cartan geometry on an open subset $U \subset M$. To get Theorem 3, it still remains to show that U is in fact the whole M, or M minus a point. This is done thanks to a rigidity result for geometrical embeddings of certain Cartan geometries, of independant interest. This result is stated at the end of section 6 (Theorem 6), and the last part of the article is devoted to its proof.

2 Geometry of the model spaces X

2.1 Algebraic preliminaries

Most of the following preliminaries are very clearly exposed in [K].

Let G be a simple Lie group of real rank 1, with finite center , and $\mathfrak{g}$ its Lie algebra. We choose a Cartan involution θ on $\mathfrak{g}$. This involution determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k}_\theta \oplus \mathfrak{p}_\theta$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}_\theta$. Since we supposed the rank of G to be 1, the dimension of $\mathfrak{a}$ is also 1. Let Δ be the set of roots for the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$. For any $\lambda \in \Delta$, the space $\mathfrak{g}_\lambda$ is defined as $\mathfrak{g}_\lambda = \{v \in \mathfrak{g} \mid Ad(a^t)v = e^{t\lambda(X_0)}v\}$. Since the rank of G is one, there are two possibilities for Δ :

- $\Delta = \{-\alpha, +\alpha\}$ (case $\mathfrak{g} = \mathfrak{so}(1,n)$, $n \geq 2$, or $\mathfrak{su}(1,1)$ or $\mathfrak{sp}(1,1)$).

- $\Delta = \{-2\alpha, -\alpha, +\alpha, +2\alpha\}$ (all the other cases).

We choose $X_0 \neq 0$ in $\mathfrak{a}$ such that $\alpha(X_0) = 1$, and denote by A the one parameter subgroup $a^t = Exp_G(tX_0)$.

In any of the two cases, the Lie algebra $\mathfrak{g}$ admits the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$. The algebra $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. It is stable under the action of the involution θ, and $\mathfrak{a}$ is the eigenspace of $\mathfrak{l}$ associated to the eigenvalue -1, since $\mathfrak{m}$ is just the space of fixed points of θ on $\mathfrak{l}$.

One can write $\mathfrak{n}^+ = \mathfrak{n}_1^+ \oplus \mathfrak{z}^+$ (resp. $\mathfrak{n}^- = \mathfrak{z}^- \oplus \mathfrak{n}_1^-$) where $\mathfrak{z}^+$ (resp. $\mathfrak{z}^-$) is the center of $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$).

Precisely, if $\Delta = \{-\alpha, +\alpha\}$, we simply have $\mathfrak{z}^+ = \mathfrak{n}^+ = \mathfrak{g}_{+\alpha}$ (resp. $\mathfrak{z}^- = \mathfrak{n}^- = \mathfrak{g}_{-\alpha}$), and $\mathfrak{n}_1^+ = \mathfrak{n}_1^- = \{0\}$.

When $\Delta = \{-2\alpha, -\alpha, +\alpha, +2\alpha\}$, then $\mathfrak{n}^+ = \mathfrak{g}_{+\alpha} \oplus \mathfrak{g}_{+2\alpha}$ (resp. $\mathfrak{n}^- = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha}$), $\mathfrak{z}^+ = \mathfrak{g}_{+2\alpha}$ (resp. $\mathfrak{z}^- = \mathfrak{g}_{-2\alpha}$)) and $\mathfrak{n}_1^+ = \mathfrak{g}_{+\alpha}$ (resp. $\mathfrak{n}_1^- = \mathfrak{g}_{-\alpha}$).

The two Lie algebras $\mathfrak{n}^+$ et $\mathfrak{n}^-$ are nilpotent. The exponential Exp_G is a diffeomorphism from $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) onto a closed subgroup $N^+ \subset G$ (resp. $N^- \subset G$).

The parabolic subgroup P is the closed subgroup of G with Lie algebra $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$. We denote π_X the projection $G \to G/P = \mathbf{X}$. As we already said in the introduction, the space $\mathbf{X}$ is diffeomorphic to a sphere.

Let us also recall the *Langland's decomposition* $P = MAN^+$ for the group P, where M and A are closed subgroups of G, with respective Lie algebras $\mathfrak{m}$ et $\mathfrak{a}$. We will denote $L = MA$.

2.2 Charts

- *Atlas :* Let us call o the projection of P on G/P. As the rank of G is one, its Weyl group is reduced to two elements: $W(G) = \{e, w\}$. The element w acts on $\mathbf{X}$ by an involution, and sends o to a point $\nu \in \mathbf{X}$. The points o and ν are left fixed by the group L. Also, $wN^+w = N^-$. The Bruhat decomposition (see [K] for example) writes $G = P \cup wN^+wP = P \cup N^-P$. In other words, if one calls Ω_o (resp. Ω_ν) the orbit of o under the action of N^- (resp. the orbit of ν under the action of N^+), the manifold $\mathbf{X}$ can be written as the union : $\mathbf{X} = \{o\} \cup \Omega_\nu = \{\nu\} \cup \Omega_o$.

This gives an atlas with two charts on $\mathbf{X}$: the mapping $s^- : \mathfrak{n}^- \to \mathbf{X}$ defined by $s^-(u) = Exp_G(u).o$, which is a diffeomorphism from $\mathfrak{n}^-$ onto the open set Ω_o. We also have the chart $s^+ : \mathfrak{n}^+ \to \mathbf{X}$ defined by $s^+(u) = Exp_G(u).\nu$, which maps $\mathfrak{n}^+$ diffeomorphically onto Ω_ν.

- *Auxiliary metrics:* we endow $\mathfrak{g}$ with a scalar product $< >_{\mathfrak{g}}$, invariant by the Cartan involution θ. We denote by $||.||$ the associated norm on $\mathfrak{g}$. We

extend this scalar product to a left invariant Riemannian metric ρ_G on G. This metric induces on N^+ and N^- two left invariant Riemannian metrics ρ^+ and ρ^-, that we carry on Ω_ν and Ω_o. We thus get two Riemannian metrics on Ω_ν and Ω_o (that we still denote by ρ^+ and ρ^-), for which the actions of N^+ and N^- are isometric.

- *Change of charts* : We denote $s_-^+ = (s^+)^{-1} \circ s^-$. The map s_-^+ maps $\mathfrak{n}^- \setminus \{0\}$ on $\mathfrak{n}^+ \setminus \{0\}$. We are going to give a formula for the restriction of s_-^+ to $\mathfrak{z}^- \setminus \{0\}$.

Let $u \in \mathfrak{z}^-$, $u \neq 0$, and $w = [u, \theta u]$. Since we saw that there is $\lambda \in \Delta$ such that $\mathfrak{z}^- = \mathfrak{g}_\lambda$, we get that $\mathbb{R}.u \oplus \mathbb{R}.w \oplus \mathbb{R}.\theta u$ is a subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{s}l(2, \mathbb{R})$. In fact, we can choose a normalization $u' = \lambda u$, $w' = \mu w$, and $v' = \theta u$, such that $[w', u'] = 2u'$, $[w', v'] = -2v'$ and $[u', v'] = w'$. The formula of the stereographic projection in dimension one yields a real a_u (which depends only of the direction of u) such that :

$$s_-^+(u) = a_u . \frac{\theta u}{||u||^2} \tag{2}$$

2.3 Geodesics on X

We begin with some notations. For $u \in \mathfrak{g}$, we call u^* the curve from $[0, 1]$ to $\mathbf{X}$ defined by $u^*(t) = \pi_X \circ Exp_G(tu)$. By $[u]$, we will mean the geometrical arc supporting u^*.

2.3.1 Parametrized geodesics

We define the following subset $\mathbf{Q} \subset \mathfrak{g}$:

$$\mathbf{Q} = \{v \in \mathfrak{g} \mid v = Ad(b).u, \ u \in \mathfrak{z}^-, \ b \in P\}$$

Definition 1 (Parametrized geodesics and geodesic segments). *We call geodesic of* $\mathbf{X}$ *any curve from* I *to* $\mathbf{X}$*, where* I *is an interval of* $\mathbb{R}$ *containing* 0*, which is of the form* $t \to g.\pi_X \circ Exp_G(tu)$ *with* $u \in \mathbf{Q}$ *and* $g \in G$*. When* $I = \mathbb{R}$*, we speak of maximal geodesic.*

- For $u \in \mathbf{Q}$*, the curve* u^* *is called the parametrized geodesic segment with origin* o *associated to* u*.*

- One says that $[u]$ *is the (unparametrized) geodesic segment with origin* o*, associated to* u*. The space of geodesic segments with origin* o *is denoted by* $[\mathbf{Q}]$*.*

- We will denote by $\dot{\mathbf{Q}}$ *(resp.* $[\dot{\mathbf{Q}}]$*) the space* $\mathbf{Q}$ *with* $0_\mathfrak{g}$ *removed (resp. the space* $[\mathbf{Q}]$ *with the "trivial" segment* $[o]$ *removed).*

In all what follows, we will endow $[\mathbf{Q}]$ with the Hausdorff topology on closed sets of $\mathbf{X}$.

2.3.2 Fundamental properties of $[\mathbf{Q}]$

Our first task is to understand the action of P on $\mathbf{Q}$, and for that, we have to desribe $[\mathbf{Q}]$ more precisely.

Lemma 1. *There is a morphism $\rho : P \to Aff(\mathfrak{n}^+)$ such that for all $b \in P$: $b \circ s^+ = s^+ \circ \rho(b)$.*

One looks separately at the action of the different components of Langland's decomposition.

• *Action of L.*

Let l be an element of L, and $u \in \mathfrak{n}^+$. We set $n^+ = Exp_G(u)$ and $x = n^+.\nu$. Then $l.x = Ad(l)(n^+).l.\nu$, and since $l.\nu = \nu$, $l.x = s^+(Ad(l).u)$. So, $\rho(l) = Ad(l)_{|\mathfrak{n}^+}$: the action of L in the chart s^+ is just the adjoint action.

• *Action of N^+.*

Let $n_0^+ = Exp(u_0)$ and $n^+ = Exp(u)$ be two elements of N^+. Since $\mathfrak{n}^+$ is a one, or two-step nilpotent Lie algebra, the Campbell-Hausdorff formula yields $n_0^+ n^+ = Exp(u_0 + u + \frac{1}{2}ad(u_0)(u))$. We thus get that $n_0^+.s^+(u) = s^+((Id + \frac{1}{2}ad(u_0)).u + u_0)$, or :

$$\rho(n_0^+) : u \mapsto (Id + \frac{1}{2}ad(u_0)).u + u_0$$

Remark 1. *If $u \in \mathfrak{z}^+$, $\rho(n_0^+).u = u + u_0$, so that $\rho(N^+)$ acts by translations on $\mathfrak{z}^+$.*

For every half-line α of $\mathfrak{n}^+$, there is a unique point $x \in \mathfrak{n}^+$ and a unique vector $v \in \mathfrak{n}^+$ of norm 1 for $||.||$, such that $\alpha = \{y \in \mathfrak{n}^+ \mid y = x+tv, \ , t \in \mathbb{R}_+\}$. So, if $S_{\mathfrak{n}^+}$ (resp. $S_{\mathfrak{z}^+}$) is the unit sphere of $\mathfrak{n}^+$ (resp. $\mathfrak{z}^+$) for the norm $||.||$, the space of half-lines of $\mathfrak{n}^+$ (resp. of half-lines whose direction is in $\mathfrak{z}^+$) is identified with the product $\mathfrak{n}^+ \times S_{\mathfrak{n}^+}$ (resp. $\mathfrak{n}^+ \times S_{\mathfrak{z}^+}$). In the following, such an half-line will be denoted by $[x, u)$, with $x \in \mathfrak{n}^+$ and $u \in S_{\mathfrak{n}^+}$ (resp. $u \in S_{\mathfrak{z}^+}$).

Remark 2. *The group $\rho(P)$ leaves globally invariant the set of half-lines of $\mathfrak{n}^+$ whose direction is in $\mathfrak{z}^+$.*

Proposition 1.
(i) Under the map $\mu : [u] \to (s^+)^{-1}([u] \cap \Omega_\nu)$*, the space* $[\dot{\mathbf{Q}}]$ *is homeomorphic to the space of affine half-lines of* $\mathfrak{n}^+$ *whose direction is in* $\mathfrak{z}^+$*.*
(ii) There exists a continuous section $s : [\dot{\mathbf{Q}}] \to \dot{\mathbf{Q}}$*.*

Proof:

Let us first remark that for all $b \in P$, the following equivariance relation is true :

$$b.[u] = \rho(b).\mu([u])$$

Then, the formula (2) ensures that if $u \in \mathfrak{z}^-$ then $\mu([u])$ is the half-line $[a_u \frac{\theta u}{||u||^2}, \frac{\theta u}{||u||})$. Now, every $u' \in \mathbf{Q}$ writes $Ad(b).u$ for $b \in P$ and $u \in \mathfrak{z}^-$. From the relation $b.[u] = \rho(b).\mu([u])$ and the remark 2, one infers that the image of μ is included in the set of affine half-lines of $\mathfrak{n}^+$, whose direction is in $\mathfrak{z}^+$. Since μ is clearly an homeomorphism on its image, we only have to show that μ is surjective to conclude the proof. From the formula (2), it is clear that any half-line $[u, u)$, with $u \in S_{\mathfrak{z}^+}$, is in the image of μ. Then, we make $\rho(N^+)$ act on these half-line, and we get that all the half-lines of $\mathfrak{n}^+$, whose direction is in $\mathfrak{z}^+$ are in the image of μ.

We still have to define the section s. We define first $\tilde{s} : \mathfrak{n}^+ \times S_{\mathfrak{z}^+} \to \dot{\mathbf{Q}}$ by : $\tilde{s} : [x, v) \mapsto Ad(Exp_G(x - v)).\frac{\theta v}{a_{\theta v}}$

Then, we set $s = \tilde{s} \circ \mu$. Let us check that s is really a section:

$[\tilde{s}([x, v))] = Exp_G(x-v).[\frac{\theta v}{a_{\theta v}}] = \mu^{-1}(\rho(Exp_G(x-v)).[v, v)) = \mu^{-1}([x, v))$.

□

Lemma 2. *The space* $[\mathbf{Q}]$ *has the following properties :*
(P_1) *For every point* $x \in \mathbf{X}$*, there is a segment* $\alpha \in [\mathbf{Q}]$ *joining* o *and* x*.*
(P_2) *For the Hausdorff topology,* $[\mathbf{Q}]$ *is a closed subset of* $\mathcal{K}(\Omega_o)$*, the set of compact subsets of* Ω_o*.*
(P_3) *Let* (α_k) *be a sequence of* $[\mathbf{Q}]$ *tending to* $[o]$*, then* $\lim_{k \to +\infty} L^-(\alpha_k) = 0$*, where* $L^-(\alpha_k)$ *is the length of the segment* α_k *for the metric* ρ^-*.*

Proof:

Property (P_1) is trivial if $x = o$. If it is not the case, $x \in \Omega_\nu$. We set $y = (s^+)^{-1}(x)$ and we choose $u \in S_{\mathfrak{z}^+}$. Then $\mu^{-1}([y, u))$ is a segment of $[\mathbf{Q}]$ joining o and x.

Property (P_2) follows from the homeomorphism μ between $[\dot{\mathbf{Q}}]$ and the set of half-lines of $\mathfrak{n}^+$ whose direction is in $\mathfrak{z}^+$, and the fact that, up to subsequence, any diverging sequence of $\mathfrak{n}^+ \times S_{\mathfrak{z}^+}$ either tends to $[o]$, or leaves every compact subset of $\mathcal{K}(\Omega_o)$.

We now prove property (P_3). Let us suppose tat (P_3) is not true, and suppose the existence of a sequence $\alpha_k \in [\dot{\mathbf{Q}}]$ which tends to $[o]$, and such that $L^-(\alpha_k) > \epsilon > 0$, for all $k \in \mathbf{N}$. We identify $[\dot{\mathbf{Q}}]$ with the space of half-lines in $\mathfrak{n}^+$ whose direction is in $\mathfrak{z}^+$, so that we note $\alpha_k = [x_k, u_k)$, with $x_k \in \mathfrak{n}^+$ and $u_k \in S_{\mathfrak{z}^+}$. Looking if necessary at a subsequence, we will suppose that u_k has a limit $u_\infty \in S_{\mathfrak{z}^+}$. In the proof, we will use the following criteria, whose proof is easy:

Lemma 3. *Let $[x_k, u_k)$ be a sequence of $[\dot{\mathbf{Q}}]$. We suppose that $x_k \to \infty$ (i.e (x_k) leaves every compact subset of $\mathfrak{n}^+$), and that there are u_∞ and v_∞ in $S_{\mathfrak{z}^+}$ such that $\lim_{k\to+\infty} u_k = u_\infty$ and $\lim_{k\to+\infty} \frac{x_k}{||x_k||} = v_\infty$. Then the sequence $[x_k, u_k)$ tends to $[o]$ if and only if $v_\infty \neq -u_\infty$.*

Let a^t be the Cartan flow introduced in section 2.1. The flow a^t acts on $\mathfrak{n}^-$ by transformations of the form

$$\begin{pmatrix} e^{-2t} Id_{\mathfrak{z}^-} & 0 \\ 0 & e^{-t} Id_{\mathfrak{n}_1^-} \end{pmatrix}$$

So, the norm associated to any scalar product on $\mathfrak{n}^-$ is contracted exponentially by $ad(a^t)$ (when $t \geq 0$). As a consequence, a^t acts also by contractions for the metric ρ^- : there is a constant $c > 0$ such that $L^-(a^t.[\sigma]) \leq e^{-ct} L^-([\sigma])$.

For any sequence (x_k) tending to infinity in $\mathfrak{n}^+$ there is a sequence (t_k) of reals, with $\lim_{k\to+\infty} t_k = -\infty$ such that $Ad(a^{t_k}).x_k$ remains in a compact subset of $\mathfrak{n}^+ \setminus \{0_{\mathfrak{n}^+}\}$. Looking if necessary at a subsequence, we will suppose that $\sigma_k = a^{t_k}.\alpha_k$ tends to $\sigma_\infty = [x_\infty, u_\infty)$, with $x_\infty \neq 0_{\mathfrak{n}^+}$.

If there is no $\mu > 0$ such that $x_\infty = -\mu u_\infty$, the segment σ_∞ does not contain $0_{\mathfrak{n}^+}$, and thus $L^-([\sigma_k])$ is bounded from above by a real M, for any $k \in \mathbf{N}$. This yields a majoration of the form $L^-(\alpha_k) \leq M e^{ct_k}$, which contradicts the hypothesis $L^-(\alpha_k) > \epsilon$ (because $t_k \to -\infty$).

Now, if there exists $\mu > 0$ such that $x_\infty = -\mu u_\infty$. We write $x_k = y_k + z_k$, with $y_k \in \mathfrak{n}_1^+$ and $z_k \in \mathfrak{z}^+$. One has $\lim_{k\to+\infty} e^{t_k} y_k = 0$ and $\lim_{k\to+\infty} e^{2t_k} z_k = x_\infty = -\mu u_\infty$. So, $\lim_{k\to+\infty} e^{2t_k}(y_k + z_k) = -\mu u_\infty$, which implies $\lim_{k\to+\infty} \frac{x_k}{||x_k||} = -u_\infty$. But lemma 3 then ensures that the sequence $[x_k, u_k)$ did not tend to $[o]$, yielding a contradiction.

$\square$

2.4 Dynamical aspects

2.5 Some general definitions

We now introduce some dynamical notions which will be used all along the article.

Let M be a manifold, and G a subgroup of homeomorphisms of M.

Definition 2 (Stable data). *We call stable data a quadruple $(f_k, x_k, x_\infty, y_\infty)$, where (f_k) is a sequence of G, (x_k) is a sequence of M converging to $x_\infty \in M$, and such that the sequence $y_k = f_k(x_k)$ converges to $y_\infty \in M$.*

It is easy to check that a subgroup $G \subset Homeo(M)$ does not act properly (i.e has a sequence which does not act properly) if and only if there exists a stable data $(f_k, x_k, x_\infty, y_\infty)$, where (f_k) is a sequence of G tending to infinity in $Homeo(M)$ (we say that a sequence tends to infty in $Homeo(M)$ if its intersection with any compact subset of $Homeo(M)$ is finite).

Definition 3 (Equicontinuity). *Let $(f_k, x_k, x_\infty, y_\infty)$ be a stable data of G. One says that the action of (f_k) is equicontinuous at x_∞ if there exists a subsequence (f'_k) of (f_k), such that for any sequence x'_k tending to x_∞, $f'_k(x'_k)$ tends to y_∞.*

2.5.1 North-South dynamics on X

Lemma 4. *Let (g_k) be a sequence of G tending to infinity. Then, looking if necessary at a subsequence of (g_k), there exist two points o^+ and o^- of $\mathbf{X}$ (which can be the same), such that (g_k) has the following dynamical properties:*
(i) For every compact subset $K \subset \Omega_{o^+} = \mathbf{X} \setminus \{o^-\}$, $\lim_{k\to+\infty} g_k(K) = o^+$.
(ii) For every compact subset $K \subset \Omega_{o^-} = \mathbf{X}\setminus\{o^+\}$, $\lim_{k\to+\infty}(g_k)^{-1}(K) = o^-$.

Proof: The group G admits a *Cartan's decomposition* $G = KAK$, where K is the maximal compact subgroup of G. Thus, it is sufficient to do the proof of the lemma for a sequence $g_k = a_k$ of elements of A, tending to infinity. Considering if necessary a_k instead of a_k^{-1}, we get $a_k = a^{t_k}$, with $t_k \to +\infty$. The matricial expression of the action of $Ad(a^t)$ on $\mathfrak{n}^-$, given in the proof of lemma 2 shows that for any compact subset K of $\mathfrak{n}^-$, $Ad(a^t).K$ tends uniformly to 0 as $t \to +\infty$. Now, from the relation $s^-(Ad(a^{t_k}).u) = a^{t_k}.s^-(u)$, we get the point (i) of the lemma, setting $o^+ = s^-(0)$. The point (ii) is proved doing the same work on a^{-t_k}, acting on $\mathfrak{n}^+$, and setting $o^- = s^+(0)$. $\square$

As a consequence of lemma 4 we get the:

Corollary 1. *Let (b_k) be a sequence of P tending to infinity, with poles o^+ and o^-. The action of (b_k) is equicontinuous at o if and only if $o = o^+$, and $o^- \neq o$. In this case, there exist $l_{1,k}$ and $l_{2,k}$ two sequences of P, relatively compact in P, such that $b_k = l_{1,k} a_k l_{2,k}$, where $a_k \in A$.*

Proof: If (b_k) is a sequence of P tending to infinity, the dynamical study made in the lemma proves that the only possible fixed points for (b_k) are o^+ and o^-. Also, the action of (b_k) is never equicontinuous at o^- (but it is equicontinuous on the whole Ω_{o^+}). We infer that $o = o^+$, and $o^- \neq o^+$. Now, thanks to the previous lemma, $\lim_{k\to+\infty} b_k^{-1}.\nu = o^-$. Since $o^- \in \Omega_\nu$, there is a sequence n_k in N^+, relatively compact in N^+, so that $n_k g_k^{-1}.\nu = \nu$. The sequence $n_k g_k^{-1}$ fixes o and ν, so that it is a sequence of $L = AM$. We deduce the existence of a sequence m_k of M such that $n_k g_k^{-1} m_k$ is in A, which concludes the proof. □

2.5.2 Dynamics of P on the space $[\mathbf{Q}]$

Proposition 2. *Let (b_k)be a sequence of P which tends to infinity. Then, considering if necessary (b_k^{-1}) instead of (b_k), and looking if necessary at a subsequence:*

(i) There is an open set $\mathbf{\Omega}^+ \subset [\dot{\mathbf{Q}}]$, containing $[o]$ in its closure, such that for any compact subset K of $\mathbf{\Omega}^+$, $b_k.K$ tends to $[o]$ as k tends to $+\infty$.

(ii) The closure of the set $s(\Omega^+)$ contains elements of $\mathfrak{z}^- \setminus \{0_\mathfrak{g}\}$ which are arbitrarily close to $0_\mathfrak{g}$.

Proof: The point (ii) is just a technical point that will be useful in section 6.

In the whole proof, we identify $[\dot{\mathbf{Q}}]$ with $\mathfrak{n}^+ \times S_{\mathfrak{z}^+}$ thanks to the map μ. We don't change the conclusions of Proposition 2 if we compose (b_k) by a sequence of the compact group $M \subset P$ (observe that $Ad(M)$ preserves $\mathfrak{z}^-$). So, we will suppose that (b_k) is a sequence of $AN^+ \subset P$, and we write $b_k = a^{t_k} n_k^+$. On $\mathfrak{n}^+$, $\rho(b_k) = L_k + T_k$, namely $\rho(b_k)$ is the composition of the linear map L_k and the translation of vector T_k. In a basis compatible with the grading $\mathfrak{n}^+ = \mathfrak{n}_1^+ \oplus \mathfrak{z}^+$, the matrix of L_k is of the form $\begin{pmatrix} e^{t_k} & 0 \\ D_k & e^{2t_k} \end{pmatrix}$:

Considering if necessary a subsequence of (b_k), and (b_k^{-1}) instead of (b_k), we will suppose that t_k has a limit in $\mathbf{R}_+^* \cup \{+\infty\}$. So, looking at a new subsequence of (b_k), there exist two sequences of $\mathbb{R}^+$, β_k and μ_k, tending to μ_∞ and β_∞ in $\mathbb{R}^+ \cup \{+\infty\}$ and $\mathbb{R}_+^* \cup \{+\infty\}$ respectively, as well as a sequence B_k (resp. τ_k) in $End(\mathfrak{n}^+)$ (resp. in $S_{\mathfrak{n}^+}$) converging to $B_\infty \neq 0$ in $End(\mathfrak{n}^+)$ (resp. $\tau_\infty \in S_{\mathfrak{n}^+}$), such that $\rho(b_k) = \beta_k B_k + \mu_k \tau_k$, and such that we are in one of the three following cases :

• *First case :* $\lim_{k\to+\infty} \frac{\beta_k}{\mu_k} = 0$.
We set:

$$F = \{[x,u) \in [\dot{\mathbf{Q}}] \mid u = -\tau_\infty\}$$

This is a closed subset of $[\dot{\mathbf{Q}}]$ (which is empty if $\tau_\infty \notin S_{\mathfrak{z}^+}$). Let $[x_k, u_k)$ be a sequence of $\boldsymbol{\Omega}^+ = [\dot{\mathbf{Q}}] \setminus F$ converging to $[x_\infty, u_\infty) \in \boldsymbol{\Omega}^+$. One has $\rho(b_k)[x_k, u_k) = [x'_k, u_k)$, with $x'_k = \mu_k(\frac{\beta_k}{\mu_k} B_k.x_k + \tau_k)$. Under our hypothesis $\mu_k \to +\infty$ and thus $\lim_{k\to+\infty} \frac{x'_k}{||x'_k||} = \tau_\infty$. Since $u_\infty \neq -\tau_\infty$, Lemma 3 applies and we get that $\lim_{k\to+\infty} \rho(b_k)[x_k, u_k) = [o]$.

If $x \in \mathfrak{z}^-$, we observe that $[\theta x, \frac{\theta x}{||x||})$ and $[-\theta x, \frac{-\theta x}{||x||})$ can't be in F simultaneously. This proves that $s(\boldsymbol{\Omega}^+)$ contains some elements of $\mathfrak{z}^-$, which are arbitrarly close to $0_{\mathfrak{g}}$. This imply the point (ii) in this case.

• *Second case :* $\lim_{k\to+\infty} \frac{\beta_k}{\mu_k} = +\infty$.
Let us set:

$$F = \{[x,u) \in [\dot{\mathbf{Q}}] \mid B_\infty.x \in \mathbb{R}^-.u\}$$

This is a closed subset of $[\dot{\mathbf{Q}}]$. We first observe that in this case, we have necessarily $\beta_\infty = +\infty$. If it were not the case, we should have $\mu_\infty = 0$, and $\beta_\infty \in \mathbb{R}^*_+$. In this case L_k would converge to $L_\infty \in GL(\mathfrak{g})$, and the sequence (b_k) would be bounded.

Let $[x_k, u_k)$ be a sequence of $\boldsymbol{\Omega}^+ = [\dot{\mathbf{Q}}] \setminus F$ converging to $[x_\infty, u_\infty) \in \boldsymbol{\Omega}^+$. One has $\rho(b_k)[x_k, u_k) = [x'_k, u_k)$, with $x'_k = \beta_k(B_k.x_k + \frac{\mu_k}{\beta_k}\tau_k)$, and $\lim_{k\to+\infty} \frac{x'_k}{||x'_k||} = \frac{B_\infty.x_\infty}{||B_\infty.x_\infty||}$. By definition of F, $B_\infty x_\infty \notin \mathbb{R}_-.u_\infty$. Lemma 3 applies, and we get $\lim_{k\to+\infty} \rho(b_k)[x_k, u_k) = [o]$.

As we already noticed, $\beta_\infty = +\infty$. In a basis compatible with the graduation $\mathfrak{n}^+ = \mathfrak{n}^+_1 \oplus \mathfrak{z}^+$, the matrix B_∞ has either the form $\begin{pmatrix} 0 & 0 \\ A_\infty & 0 \end{pmatrix}$, with $A_\infty \neq 0$ (this possibility occures only if $\mathfrak{g} \neq \mathfrak{so}(1,n), \mathfrak{su}(1,1)$ or $\mathfrak{sp}(1,1)$), or of the form $\begin{pmatrix} 0 & 0 \\ C_\infty & Id_{\mathfrak{z}^+} \end{pmatrix}$ (we can have $C_\infty = 0$).

In this last case, if $x \in \mathfrak{z}^- \setminus \{0_{\mathfrak{g}}\}$, then $\theta x \notin \mathbb{R}^-.\frac{\theta x}{||x||}$. As a consequence, $\mathfrak{z}^- \setminus \{0_{\mathfrak{g}}\} \subset s(\boldsymbol{\Omega}^+)$ and the point (ii) is proved in this case.

In the first case, we pick a sequence y_n in $\mathfrak{n}^+_1$ which tends to $0_{\mathfrak{g}}$, and such that for every $n \in \mathbf{N}$, $A_\infty.y_n \notin \mathbb{R}_-.\frac{\theta x}{||x||}$. Then for $x \in \mathfrak{z}^- \setminus \{0_{\mathfrak{g}}\}$, $s([\theta x + y_n, \frac{\theta x}{||x||}))$ is a sequence of $s(\boldsymbol{\Omega}^+)$ which tends to x, what proves the point (ii).

• *Third case :* $\lim_{k\to+\infty} \frac{\beta_k}{\mu_k} = \alpha$*, with* $\alpha \in \mathbb{R}^*_+$.

Let us observe, as in the previous case, that since (b_k) tends to infinity, we have necessarily $\beta_\infty = \mu_\infty = +\infty$ in this case.

We set:

$$F = \{[x, u) \in [\dot{\mathbf{Q}}] \mid \alpha B_\infty.x + \tau_\infty \in \mathbb{R}^-.u\}$$

This is a closed subset of $[\dot{\mathbf{Q}}]$. Let $[x_k, u_k)$ be a sequence of $\mathbf{\Omega}^+ = [\dot{\mathbf{Q}}] \setminus F$ converging to $[x_\infty, u_\infty) \in \mathbf{\Omega}^+$. One has $\rho(b_k)[x_k, u_k) = [x'_k, u_k)$, and $x'_k = \mu_k(\frac{\beta_k}{\mu_k} B_k.x_k + \tau_k)$. Since $[x_\infty, u_\infty) \in [\dot{\mathbf{Q}}] \setminus F$, we get $\alpha B_\infty.x_\infty + \tau_\infty \neq 0$, so that $\lim_{k\to+\infty} \frac{x'_k}{||x'_k||} = \frac{\alpha B_\infty.x_\infty + \tau_\infty}{||\alpha B_\infty.x_\infty + \tau_\infty||}$. Since $\alpha B_\infty.x_\infty + \tau_\infty \notin \mathbb{R}_-.u_\infty$, we can once again apply Lemma 3, and we conclude: $\lim_{k\to+\infty} \rho(b_k)[x_k, u_k) = [o]$.

Since we are still in the case $\beta_\infty = +\infty$, the possible matrices for B_∞ are the same as in the previous case. One thus checks that when $x \in \mathfrak{z}^-$ is very close to $0_{\mathfrak{g}}$, $[\theta x, \frac{\theta x}{||x||})$ and $[-\theta x, -\frac{\theta x}{||x||})$ can't be in F simultaneously, what proves point (ii).

$\square$

3 Cartan geometries

Here, we just indroduce the basic material on Cartan's geometries. For more details, and the proofs of some of the lemmas and propositions below, we refer to the very good reference [Sha]. In all this section, we consider a Cartan geometry (M, B, ω), modelled on $\mathbf{X} = G/P$ (for the following generalities, we don't make special asumptions, so that G is any Lie group, and P a closed subgroup of G).

3.1 Parallelism and Riemannian metric on B

Let $(E_1, ..., E_s)$ be a basis of the Lie algebra $\mathfrak{g}$, which is orthonormal for the metric $< >_{\mathfrak{g}}$. At every point p of B, we define a frame $\mathcal{R}(p) = (E_1^\dagger(p), ..., E_s^\dagger(p))$ by $E_k^\dagger(p) = \omega_p^{-1}(E_k)$. This frame field yields a parallelism $\mathcal{R}$ on B. For any vector $\xi = \Sigma\lambda_k E_k$ in $\mathfrak{g}$, we define a vector field $\xi^\dagger$ on B by $\xi^\dagger(p) = \Sigma\lambda_k E_k^\dagger(p)$. Such vector fields with constant coordinates with respect to $\mathcal{R}$ are called *parallel.*

Notation 1. *We will adopt the notation ξ_p instead of $\xi^\dagger(p)$. Also, if $\mathbf{\Lambda}$ is a subset of $\mathfrak{g}$, it determines a field of "parallel subsets" of TB, defined by $\Lambda_p = \omega_p^{-1}(\mathbf{\Lambda})$, for every $p \in B$. For a sequence $\mathbf{\Lambda}_k$ of subsets, we will write $\Lambda_{p,k} = \omega_p^{-1}(\mathbf{\Lambda}_k)$.*

If $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, $H_{p,\mathfrak{h}}$ denotes $\omega_p^{-1}(\mathfrak{h})$.

Definition 4. *We call ρ the Riemannian metric on B for which $\mathcal{R}(p)$ is an orthonormal frame for all $p \in B$.*

Let us notice that for all $p \in B$, $\rho_p = \omega_p^*(< >_{\mathfrak{g}})$.

3.2 Curvature

The Maurer-Cartan form on the Lie group G satisfies the so called *structure equation* : $d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$. Nevertheless, for a given Cartan connection ω, the 2-form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ needs not to be trivial. One calls Ω *the curvature* of the connection ω. Here are some fundamental properties of the curvature form :

(i) For every $b \in P$, $(R_b)^*\Omega = Ad(b^{-1})\Omega$.

(ii) $\Omega(X, Y) = 0$ if X or Y is a vertical vector (i.e tangent to the fibers of $B \to M$).

A Cartan geometry (M, B, ω) is said to be *flat* if Ω vanishes identically on B.

Let us notice that the equivariance property (i) implies that the vanishing of the curvature at a point of a fiber implies its vanishing on the whole fiber. In the following, we will sometimes say abusively that ω vanishes at $x \in M$, meaning that ω vanishes on the fiber over x.

Example 1. *Let U be an open subset of $\mathbf{X}$, stable under the action of a discrete subgroup $\Gamma \subset G$. If the action of Γ on U is free and properly discontinuous, then the manifold $M = \Gamma \backslash U$ is naturally endowed with a flat Cartan geometry, inherited from that of $\mathbf{X}$. The bundle B is, in this case, the quotient $\Gamma \backslash G$ and the Cartan connection is $\overline{\omega}_G$, the 1-form induced by ω_G on $\Gamma \backslash G$.*

3.2.1 Regularity

We now say a few words on the asumption made on the connection ω in Theorem 3. Let us suppose that (M, B, ω) is modelled on a space $\mathbf{X} = G/P$, such that the Lie algebra $\mathfrak{g}$ is endowed with a k-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$, $k \in \mathbf{N}^*$, and the Lie algebra of P is $\mathfrak{p} = \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$. Notice that for all $-k \leq i \leq j \leq k$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. We then have an $Ad(P)$-invariant filtration $\mathfrak{g}^{-k} = \mathfrak{g} \supset \mathfrak{g}^{-k+1} \supset ... \supset \mathfrak{g}^k = \mathfrak{g}_k$, putting $\mathfrak{g}^i = \mathfrak{g}_i \oplus \mathfrak{g}_{i+1} \oplus ... \oplus \mathfrak{g}_k$.

Definition 5 (Regularity). *The Cartan connection ω is said to be regular, if for any $\xi \in \mathfrak{g}^i$ and $\zeta \in \mathfrak{g}^j$, $i, j < 0$, and any $p \in B$, $\Omega_p(\xi_p, \zeta_p) \in \mathfrak{g}^{i+j+1}$.*

This notion of regularity is important, since in most of the cases where the equivalence problem is solved, the "canonical" Cartan connection is regular.

Let us see more closely what is going on in the case where $\mathfrak{g}$ is a rank one simple Lie algebra.

• If the roots of $\mathfrak{g}$ are $\{-\alpha, \alpha\}$, then the regularity condition just means that the curvature Ω takes its values in $\mathfrak{p}$. One says in this case that ω is *torsionfree*.

• If the roots of $\mathfrak{g}$ are $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$, then the regularity condition means that for any pair ξ, ζ in $\mathfrak{g}^{-1} = \mathfrak{n}_1^- \oplus \mathfrak{p}$, then $\Omega_p(\xi_p, \zeta_p) \in \mathfrak{g}^{-1}$ (for one, and hence for any $p \in B$).

3.3 Developping curves

One of the fundamental properties of a Cartan connection, is that it establishes a link between parametrized curves of M passing through a given point, and curves of the model space $\mathbf{X}$.

Definition 6. *Let N be a manifold, q a point of N and $I \subset \mathbb{R}$ an open interval containing 0. We define $C^1(N, I, q)$ as the space of C^1 maps $\gamma : I \to N$, such that $\gamma(0) = q$.*

We now recall how to develop curves of M into curves on $\mathbf{X}$ thanks to the Cartan connection. The proofs can be found in [Sha] (Lemma 4.12 p 208).

Lemma 5. *Let p be a point of B, and I an open interval containing 0. Then there exists a unique map $\hat{\mathcal{D}}_p : C^1(B, I, p) \to C^1(G, I, e))$ satisfying:*

(i) For every curve $\hat{\gamma}$ of $C^1(B, I, p)$, the curve $\alpha = \hat{\mathcal{D}}_p(\hat{\gamma})$ satisfies for every $t \in I$: $\omega(\hat{\gamma}'(t)) = \omega_G(\alpha'(t))$.

(ii) If $a \in C^1(P, I, a_0)$, and if we define $R_a\hat{\gamma}$ to be the curve defined by $(R_a\hat{\gamma})(t) = R_{a(t)}\hat{\gamma}(t)$, then $\mathcal{D}_{p.a_0}(R_a\hat{\gamma}) = a_0^{-1}.(R_a\alpha)$.

Corollary 2. *Let $\gamma \in C^1(M, I, x)$ and $\hat{\gamma}_1 : I \to B$, $\hat{\gamma}_2 : I \to B$ be two lifts of γ in $C^1(B, I, p)$. Then $\hat{\mathcal{D}}_p(\hat{\gamma}_1)$ and $\hat{\mathcal{D}}_p(\hat{\gamma}_2)$ project on the same curve of $\mathbf{X}$.*

Definition 7 (Developping map). *The corollary implies that for all $x \in M$, and $\hat{x} \in B$ over x, there is a well defined map from $C^1(M, I, x)$ to $C^1(\mathbf{X}, I, o)$. This map is called developping map at x, and denoted by $\mathcal{D}_x^{\hat{x}}$. If $\gamma \in C^1(M, I, x)$, and if $[\gamma]$ is the associated geometric segment, we will write $\mathcal{D}_x^{\hat{x}}([\gamma])$ instead of $[\mathcal{D}_x^{\hat{x}}(\gamma)]$.*

3.3.1 Development for flat structures

We refer once again to [Sha] for the details concerning this section.

The vanishing of the curvature is the only obstruction for the bundle (B, ω) to be locally isomorphic to (G, ω_G).

Let $\tilde{M}$ be the universal cover of M, and $r : \tilde{M} \to M$ the covering map. There exists a covering $\tilde{r} : \tilde{B} \to B$, which is a principal P-bundle $\tilde{\pi} : \tilde{B} \to \tilde{M}$, such that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\tilde{r}} & B \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{r} & M \end{array}$$

The form $\tilde{\omega} = \tilde{r}^*\omega$ is a Cartan connection on $\tilde{B}$.

Theorem 4. *If the curvature of ω vanishes identically, then there exists a bundle morphism $\tilde{\delta} : (\tilde{B}, \omega) \to (G, \omega_G)$, which is an immersion satisfying $\tilde{\delta}^*\omega_G = \tilde{\omega}$. The map $\tilde{\delta}$ induces an immersion $\delta : \tilde{M} \to \mathbf{X}$.*

In that case, the manifold M is said to be endowed with a $(G, \mathbf{X})-$structure, and the map δ is known as a *developping map* of the structure. Let (M, B, ω) be a flat Cartan geometry, x a point of M and $\tilde{x}$ a point of $\tilde{M}$ over x. We can compose $\tilde{\delta}$ by an element of G in order to get $\delta(\tilde{x}) = o$. Let $\gamma \in C^1(M, I, x)$. Then there is a unique $\tilde{\gamma} \in C^1(\tilde{M}, I, \tilde{x})$, such that $r(\tilde{\gamma}(t)) = \gamma(t)$ for every $t \in I$. If $\hat{x}$ is the point over x such that $\tilde{\delta}(\hat{x}) = e$, then $\hat{\mathfrak{D}}_x^{\hat{x}}(\tilde{\gamma}) = \delta \circ \tilde{\gamma}$.

4 The geodesics of a rank one parabolic geometry

4.1 Exponential maps

For $p \in B$ and $\xi \in T_pB$, we define $U = \omega_p(\xi)$, and ϕ_ξ^t the local flow associated to the parallel vector field $U^\dagger$. There is a neighbourhood W_p^B of 0_p in T_pB, starshaped with respect to 0_p, such that for every $\xi \in W_p^B$, $t \mapsto \phi_\xi^t(p)$ is defined on $[0, 1]$.

Notation 2. *We will denote by W^B the open subset of TB defined by $\bigcup_{p \in B} W_p^B$.*
If $p \in B$, $\mathbf{W}_p$ is the subset of $\mathfrak{g}$ defined by $\omega_p(W_p^B)$.

Definition 8. *We define an exponential map $Exp : W^B \to B$, by $Exp_p(\xi) = \phi_\xi^1(p)$, for every $p \in B$, $\xi \in W_p^B$.*

-If $x = \pi(p)$ and $\xi \in W_p^B$, we define $Exp_x(\xi) = \pi \circ Exp_p(\xi)$.

- For $\xi \in W_p^B$, we call $\hat{\xi}^*$ (resp. ξ^*) the path of B (resp. of M), parametrized by $[0, 1]$, and defined by $\hat{\xi}^*(t) = Exp_p(t\xi)$ (resp. $\xi^*(t) = Exp_x(t\xi)$). The geometric support of $\hat{\xi}^*$ (resp. ξ^*) will be denoted by $[\hat{\xi}]$ (resp. $[\xi]$).

- We will write $[x]$ instead of $[0_p]$.

- If $\Lambda_p \subset W_p^B$, we set $[\Lambda_p] = \bigcup_{\xi \in \Lambda_p} [\xi]$. Let us notice that $Exp_{\pi(p)}(\Lambda_p) \subset [\Lambda_p]$.

4.2 Geodesics

We now suppose that (M, B, ω) is a Cartan geometry modelled on $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{K}}$.

Definition 9 (Geodesics on M). *Let I be an interval containing 0, x a point of M, $\hat{x}$ over x, and $\gamma \in C^1(M, I, x)$. One says that γ is a parametrized geodesic of M if and only if $\mathcal{D}_x^{\hat{x}}(\gamma)$ is a parametrized geodesic of $\mathbf{X}$, as defined in section 2. This definition does not depend of the choice of $\hat{x}$ over x, by $Ad(P)$-invariance of $\mathbf{Q}$*

Definition 10. *- For every $p \in B$, we define:*

$$Q_p = \{\xi \in T_p B \mid \omega_p(\xi) \in \mathbf{Q}\}$$

and $\dot{Q}_p = Q_p \setminus \{0_p\}$

- $TQ = B \times_P \mathbf{Q}$ is the subbundle of TB, the fibers of which are the Q_p's.

- For every $p \in B$, we call Q_p^{reg} the regular part of Q_p, namely the set of $u \in \dot{Q}_p \cap W_p^B$ such that the restriction of $Exp_{\pi(p)}$ to Q_p is a submersion at u.

Lemma 6. *Let x be a point of M, and $\hat{x} \in B$ overx. The geodesic segments parametrized by $[0, 1]$, starting from x, are exatly the ξ^*'s, for $\xi \in W_{\hat{x}}^B \cap Q_{\hat{x}}$.*

Proof:

By the very definition of the parallel fields, we get the relation :

$$\mathcal{D}_x^{\hat{x}}(\xi^*) = (\omega_{\hat{x}}(\xi))^* \tag{3}$$

The lemma then follows immediately. $\square$

Now comes a very important lemma, which links the behaviour of a sequence of geodesic segments in M (or sets of such sequences) with the behaviour of their developments:

Lemma 7. *Let $\hat{x}_k$ be a sequence of B converging to $\hat{x}_\infty \in B$, and $\mathbf{\Lambda}_k$ a sequence of subsets of $\mathbf{Q} \cap \mathbf{W}_{\hat{x}_k}$. Then if $\lim_{k \to +\infty} [\mathbf{\Lambda}_k] = [o]$, we also have $\lim_{k \to +\infty} [\Lambda_{\hat{x}_k, k}] = [x_\infty]$ (where, as usual, $\Lambda_{\hat{x}_k, k}$ stands for $\omega_{\hat{x}_k}^{-1}(\mathbf{\Lambda}_k)$).*

Proof: For every $p \in B$, the space $H_{p, \mathfrak{n}^-}$ (see the notation 1) is called *horizontal space* at p (notice that this distrubution is not invariant by the action of P on B, unlike the case of Ehresmann's connections).

Let $x \in M$, $\hat{x} \in B$ over x, and $\gamma \in C^1(M, I, x)$, such that $\mathcal{D}_x^{\hat{x}}(\gamma) \subset \Omega_o$. Then, there exists a unique horizontal lift γ in $C^1(B, I, \hat{x})$ (i.e with horizontal tangent vector for all $t \in I$). Indeed, let $\hat{\gamma}$ be any lift of γ in $C^1(B, I, \hat{x})$, and $\hat{\alpha} = \hat{\mathcal{D}}_{\hat{x}}(\hat{\gamma})$. Since N^- is transverse to any fiber of $G \to \mathbf{X}$ over Ω_o, there is $a \in C^1(P, I, e)$ such that $R_a\hat{\alpha}$ is a curve in $C^1(N^-, I, e)$. Thanks to Lemma 5, point (ii), the curve $R_a\hat{\gamma}$ is also horizontal.

Now, let us go back to the proof of lemma 7. Suppose first that $\mathbf{\Lambda}_k$ is just a sequence of points ξ_k in $\mathbf{Q} \cap \mathbf{W}_{\hat{x}_k}$. By the previous remark, there is an horizontal curve $\hat{\gamma}_k \in C^1(B, I, \hat{x}_k)$ over $\xi^*_{\hat{x}_k,k}$. The curve $\hat{\alpha}_k = \hat{\mathcal{D}}_{\hat{x}_k}(\hat{\gamma}_k)$ is in $C^1(N^-, I, e)$ and projects on ξ^*_k. Now, we observe that if $(a_{1,k}(t), ..., a_{n,k}(t))$ are the coordinates of $\hat{\gamma}'_k(t)$ with respect to the parallelism $\mathcal{R}$, then the coordinates of $\hat{\alpha}'_k(t)$ with respect to the parallelism defined by $E_1, ..., E_n$ on TG are also $(a_{1,k}(t), ..., a_{n,k}(t))$. We thus get that $L_\rho(\hat{\gamma}_k) = L_{\rho_G}(\hat{\alpha}_k) = L^-(\xi^*_k)$. Thanks to the hypothesis of the lemma, and property (P_3) of lemma 2, we get : $\lim_{k\to+\infty} L_\rho(\hat{\gamma}_k) = 0$. So, for k sufficiently large, $[\hat{\gamma}_k]$ is included in any ρ-ball of arbitrary small radius, and centered at $\hat{x}_\infty$. Projecting, $[\xi_{\hat{x}_k,k}]$ is included, for k large enough, in any neighbourhood of x_∞, what concludes the proof.

Now, in the general case, $[\Lambda_{\hat{x}_k,k}]$ tends to $[x_\infty]$ if and only if for any sequence σ_k of $\Lambda_{\hat{x}_k,k}$, $[\sigma_k]$ tends to $[x_\infty]$, so that the proof of the general case reduces to that of the previous situation.

□

We end the section with a technical lemma:

Lemma 8. *Let x be a point of M and $\hat{x}$ a point of B over x. There is an open subset $U \subset H_{\hat{x},\mathfrak{z}^-}$, containing $0_{\hat{x}}$, such that $U \setminus \{0_{\hat{x}}\} \subset Q^{reg}_{\hat{x}}$.*

Proof: Let us first remark that for all $\xi \in \mathfrak{z}^-$, $\mathfrak{n}^- \subset T_\xi\dot{\mathbf{Q}}$. This is quite obvious when $\mathfrak{z}^- = \mathfrak{n}^-$. In the other cases, let us fix $\zeta_1, ..., \zeta_s$ a basis of $\mathfrak{n}_1^+$. The flow $exp_G(t\xi)$ commutes with none of the $Exp_G(t\zeta)$'s, for any ζ in $\mathfrak{n}^-$. Indeed, the first flow acts on $\mathbf{X}$ with a unique fixed point which is ν. The second ones act on $\mathbf{X}$ also with a unique fixed point, which is o. As a consequence, $[\xi, \zeta_1], ..., [\xi, \zeta_s]$ is a free family of vectors, which turns out to be a basis of $\mathfrak{n}_1^-$ (indeed, we are in the case where $\mathfrak{z}^- = \mathfrak{g}_{-2\alpha}$ and $\mathfrak{n}_1^+ = \mathfrak{g}_{+\alpha}$, see section 2.1. So, $[\mathfrak{z}^-, \mathfrak{n}_1^+] \subset \mathfrak{n}_1^-$). On the other hand, since for every $1 \leq i \leq n$, $[\xi, \zeta_i] = \frac{d}{dt}_{|t=0} Ad(Exp_G(t\zeta_i)).\xi$, we get $[\xi, \zeta_i] \in T_\xi\dot{\mathbf{Q}}$. Finally, $\mathfrak{n}_1^- \subset T_\xi\dot{\mathbf{Q}}$. By definition of $\mathbf{Q}$, $\mathfrak{z}^- \subset T_\xi\dot{\mathbf{Q}}$. We then get $\mathfrak{n}^- \subset T_\xi\dot{\mathbf{Q}}$.

Let us choose an open subset $V_{\hat{x}}$ containing $0_{\hat{x}}$, such that $Exp_{\hat{x}}$ is a diffeomorphism from $V_{\hat{x}}$ onto its image. The pullback in $V_{\hat{x}}$ by $Exp_{\hat{x}}$, of the fibers of $B \to M$, yields a foliation $\mathcal{F}$ of $V_{\hat{x}}$. The leaf $\mathcal{F}_{0_{\hat{x}}}$ passing through

$0_{\hat{x}}$ is nothing else than $\omega_{\hat{x}}^{-1}(\mathfrak{p}) \cap V_{\hat{x}}$. In particular, it is transverse to $H_{\hat{x},\mathfrak{z}^-}$. Thus, there is a neighbourhood U of $0_{\hat{x}}$ in $H_{\hat{x},\mathfrak{z}^-}$, such that for every $u \in U$, $\mathcal{F}_u$ is transverse to $H_{\hat{x},\mathfrak{n}^-}$. Now, by what we said before, $H_{\hat{x},\mathfrak{n}^-} \subset T_u\dot{Q}_{\hat{x}}$ for every $u \in U$. The differential of Exp_x at u, when restricted to $T_u\dot{Q}_{\hat{x}}$ is then a surjection onto $T_{Exp_x(u)}M$.

□

5 Automorphisms of a Cartan Parabolic geometry

5.1 Preliminary remarks

In this subsection, we consider here a general Cartan geometry. In the introduction of the article, we defined the group $Aut(B,\omega)$, as the group of diffeomorphisms h in B such that $h^*\omega = \omega$. Any element of $Aut(B,\omega)$ preserves the parallelism $\mathcal{R}$, so that $Aut(B,\omega)$ is a closed subgroup of $Iso(B,\rho)$. This last group is closed in $Homeo(B)$, by classical properties of the group of isometries of a Riemannian manifold (se for example [Ko], [**?**]).

5.2 Holonomy sequences

Definition 11 (Holonomy datas). *Let $(f_k, x_k, x_\infty, y_\infty)$ be a stable data of $Aut(M,\omega)$. We say that $(b_k, \hat{x}_k, \hat{x}_\infty, \hat{y}_\infty)$ is an associated holonomy data, if there is a lift $\hat{f}_k$ of f_k in $Aut(B,\omega)$, a sequence $\hat{x}_k$ of B over x_k, tending to $\hat{x}_\infty$, and a sequence $\hat{y}_\infty \in B$ over y_∞, so that $\hat{y}_k = R_{b_k}^{-1} \circ \hat{f}_k(\hat{x}_k)$ tends to $\hat{y}_\infty$.*

To any stable data $(f_k, x_k, x_\infty, y_\infty)$ of $Aut(M,\mathcal{S})$, it is possible to associate an holonomy data $(b_k, \hat{x}_k, \hat{x}_\infty, \hat{y}_\infty)$. To see this, let us fix two open subsets U and V around x_∞ and y_∞ respectively. We suppose U and V so small that there are two continuous sections $s_U : U \to B$ and $s_V : V \to B$. Let $\hat{f}_k$ be a lift of f_k in $Aut(B,\omega)$. We then define $s_U(x_k) = \hat{x}_k$, $k \in \mathbf{N} \cup \{\infty\}$ (resp. $s_V(f_k(x_k)) = \hat{y}_k$, $k \in \mathbf{N} \cup \{\infty\}$). Since $\hat{f}_k(\hat{x}_k)$ is a sequence over $f_k(x_k)$, there is a unique sequence (b_k) of P, such that for all $k \in \mathbf{N}$, we have $R_{b_k^{-1}} \circ \hat{f}_k(\hat{x}_k) = \hat{y}_k$.

Proposition 3 (equivariance). *Let $(f_k, x_k, x_\infty, y_\infty)$ be a stable data of $Aut(M,\omega)$, and $(b_k, \hat{x}_k, \hat{x}_\infty, \hat{y}_\infty)$ an associated holonomy. For every sequence $\xi_k \in \mathbf{W}_{\hat{x}_k}$, we define $\zeta_k = Ad(b_k).\xi_k$. We then have the equivariance property:*

$$f_k(\xi^*_{\hat{x}_k,k}) = \zeta^*_{\hat{y}_k,k} \tag{4}$$

Proof: Let $\hat{f}_k$ be a lift of f_k defining the holonomy sequence b_k, and put $\phi_k = R_{b_k^{-1}} \circ \hat{f}_k$. By definition $\phi_k(\hat{x}_k) = \hat{y}_k$ and $\lim_{k\to+\infty} \hat{y}_k = \hat{y}_\infty$. We observe that $(\hat{\phi}_k)_*\xi^\dagger = (Ad(b_k)\xi)^\dagger$. We get firstly that if $\xi_k \in \mathbf{W}_{\hat{x}_k}$, then $\zeta_k \in \mathbf{W}_{\hat{y}_k}$, and secondly that $\hat{\phi}_k(\hat{\xi}^*_{\hat{x}_k,k}) = \hat{\zeta}^*_{\hat{y}_k,k}$. Projecting on M, this leads to the relation (4).

□

5.3 The rank one case

Now, we consider a Cartan geometry (M, B, ω), modelled on some space $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{K}}$.

Theorem 5.
(i) *The group $Aut(M, \omega)$ is closed in $Homeo(M)$.*

Let $(f_k, x_k, x_\infty, y_\infty)$ be a stable data of $Aut(M, \omega)$, and $(b_k, \hat{x}_k, \hat{x}_\infty, \hat{y}_\infty)$ an associated holonomy. Then:

(ii) *(f_k) is bounded in $Aut(M, \omega)$ if and only if (b_k) is bounded in G.*

(iii) *If the action of (f_k) is equicontinuous at x_∞, then the action of (b_k) is equicontinuous at o.*

Proof:

Let us begin with the proof of the point (iii).

Suppose on the contrary that (b_k) does not act equicontinuously at o. Looking if necessary at a subsequence of (b_k), one can choose a sequence (z_k) in $\mathbf{X}$, converging to o, and such that $w_k = b_k.z_k$ tends to $w_\infty \neq o$. By the property (P_1) of lemma 2, there exists, for all k, a geodesic segment $\alpha_k = [z_k, u_k)$ linking o to z_k (we still identify $[\mathbf{Q}]$ with $\mathfrak{n}^+ \times S_{\mathfrak{z}^+}$). We choose the α_k's such that $\lim_{k\to+\infty} \alpha_k = [o]$. Looking at a subsequence if necessary, we suppose that $\beta_k = b_k.\alpha_k$ tends to $\beta_\infty = [w_\infty, u_\infty)$. We put $\xi_k = s(\alpha_k)$ and $\zeta_k = s(\beta_k)$. We also set $\zeta_\infty = s(\beta_\infty)$. By continuity of s, $\lim_{k\to+\infty} \zeta_k = \zeta_\infty$. We can find a real $\lambda \in]0, 1]$ such that $\lambda\zeta_k \in \mathbf{W}_{\hat{y}_k}$ for all $k \in \mathbf{N} \cup \{\infty\}$. So, replacing if necessary ζ_k by $\lambda\zeta_k$, and ξ_k by $Ad(b_k^{-1})(\lambda\zeta_k)$, we will suppose that $\xi_k \in \mathbf{W}_{\hat{x}_k}$ for all $k \in \mathbf{N}$, and $\zeta_k \in \mathbf{W}_{\hat{y}_k}$ for all $k \in \mathbf{N} \cup \{\infty\}$.

From Proposition 3, we infer that for all $k \in \mathbf{N}$, and $t \in [0, 1]$: $f_k(\xi^*_{\hat{x}_k,k}(t)) = \zeta^*_{\hat{y}_k,k}(t)$. Since $\zeta_{\hat{y}_\infty,\infty} \neq 0$, there exists $t_0 \in [0, 1]$ such that $\zeta^*_{\hat{y}_\infty,\infty}(t_0) \neq y_\infty$. So, $\lim_{k\to+\infty} f_k(\xi^*_{\hat{x}_k,k}(t_0)) \neq y_\infty$. Nevertheless, $\xi^*_{\hat{x}_k,k}(t_0)$ tends to x_∞. Indeed, $\lim_{k\to+\infty}[\xi_k] = [o]$ and Lemma 7 implies that $\lim_{k\to+\infty}[\xi_{\hat{x}_k,k}] = [x_\infty]$. We conclude that the action of (f_k) is not equicontinuous at x_∞.

We can now prove the point (i). We suppose that (f_k) is a sequence of $Aut(M,\omega)$ converging to $f_\infty \in Homeo(M)$. Then, for all point $x \in M$, if (x_k) is the constant sequence equal to x, and $y_\infty = f_\infty(x)$, the quadruple (f_k, x_k, x, y_∞) is a stable data. Let us denote by $(b_k, \hat{x}_k, \hat{x}, \hat{y}_\infty)$ an associated holonomy data. We prove :

Lemma 9. *The sequence (b_k) is bounded in G.*

Proof: If (b_k) were unbouded, there would be a subsequence of (b_k) admitting a North-South dynamic with poles o^+ et o^-. Since o is fixed by (b_k), we have necessarily $o = o^+$ ou $o = o^-$. In the second case, the action of $(b_k)$is not equicontinuous at o (see Corollary 1), and as a consequence of point (iii), the action of (f_k) should not be equicontinuous at x. This is in contradiction with the fact that (f_k) has a limit in $Homeo(M)$. Thus, we are in the case $o = o^+$. Let $\alpha = [\xi]$ be a segment of $[\mathbf{Q}] \setminus [o]$, such that $\xi \in \mathbf{W}_{\hat{x}}$, and $\alpha \subset \Omega_o$. Let us remark that $\zeta_k = Ad(b_k).\xi \in \mathbf{W}_{\hat{y}_k}$ for all $k \in \mathbf{N}$. Since $o = o^+$, we have $\lim_{k\to+\infty} b_k.\alpha = [o]$. By Lemma 7, we obtain $\lim_{k\to+\infty}[\zeta_{\hat{y}_k,k}] = [y_\infty]$. But Proposition 3 gives that $f_k([\xi_{\hat{x}_k}]) = [\zeta_{\hat{y}_k,k}]$ for all $k \in \mathbf{N}$, which leads to $f_\infty([\xi_{\hat{x}}]) = [y_\infty]$. Since $[\xi_{\hat{x}}] \neq [x]$, we get a contradiction with the fact that $f_\infty \in Homeo(M)$.

□

We take again a subsequence of (f_k), so that (b_k) tends to $b_\infty \in P$. This means that $\hat{f}_k(\hat{x}_k)$ tends to $R_{b_\infty}.\hat{y}_\infty$. But $(\hat{f}_k)$ is a sequence of isometries for the riemannian metric ρ. Since the action of the isometry group of a riemannian manifold is proper, $\hat{f}_k$ is relatively compact in $Iso(B,\rho)$. The group $Aut(B,\omega)$ being closed in $Iso(B,\rho)$, we can suppose, looking once again at a subsequence, that there exist $\hat{f}_\infty \in Aut(B,\omega)$ such that $\hat{f}_k \to \hat{f}_\infty$. Finally, $\hat{f}_\infty$ is a lift of f_∞ in $Aut(B,\omega)$, so that $f_\infty \in Aut(M,\omega)$.

The proof of point (ii) goes in a way similar to that of point (i).

□

Remark 3. *It is likely that this use of geodesics and holonomy can be generalized, to prove that the automorphisms group of any parabolic Cartan geometry on M is closed in $Homeo(M)$.*

6 Proof of Theorem 3

We first make the proof of Theorem 3 in an easy case: the case of Kleinian manifolds (i.e manifolds M which are a quotient of some open subset $\Omega \subset \mathbf{X}$

by a discrete subgroup $\Gamma \subset G$). These manifolds are naturally endowed with a Cartan geometry modelled on $\mathbf{X}$, with $B = \Gamma \backslash \pi_X^{-1}(\Omega)$, and $\omega = \overline{\omega}_G$, the Cartan connection induced by ω_G on this quotient.

Lemma 10. *Let Ω be a connected open subset of $\mathbf{X}$, and $M = \Gamma \backslash \Omega$, where $\Gamma \subset G$ is a discrete subgroup acting freely properly discontinuously on Ω.*

If $Aut(M, \overline{\omega}_G)$ does not act properly on M, then $\Gamma = \{e\}$ and:
- *either $\Omega = \mathbf{X} \setminus \{\kappa\}$, for some $\kappa \in \mathbf{X}$.*
- *or $\Omega = \mathbf{X}$.*

Proof: The group $Aut(M, \overline{\omega}_G)$ is induced by elements of G leaving Ω stable and normalizing Γ. If $Aut(M, \overline{\omega}_G)$ does not act properly on M, we can find a sequence (h_k) of G, normalizing Γ, and acting nonproperly on Ω. We can suppose that this sequence tends to infinity in G, and looking if necessary at (h_k^{-1}) instead of (h_k), we can also asume that $o^- \in \Omega$. Let us call π_M the covering map from Ω onto M, and let $U \subset \Omega$ a small open subset containing o^-, such that π_M maps U diffeomorphically on its image. By Lemma 4, there is a sequence $(k_m)_{m \in \mathbf{N}}$ such that $h_{k_m}(U)$ is an increaszing sequence of open subsets, the union of which is the whole space $\mathbf{X}$ if $o^+ = o^-$, and $\mathbf{X} \setminus \{o^+\}$ if $o^+ \neq o^-$. We infer that $\Omega = \mathbf{X}$ or $\Omega = \mathbf{X} \setminus \{o^+\}$. Moreover, since h_k normalizes Γ for all $k \in \mathbf{N}$, π_M has to be injective on each $h_{k_m}(U)$. Finally, π_M is injective on a dense open set of Ω, which implies $\Gamma = \{e\}$.

$\square$

We now work under the general hypothesis of Theorem 3: (M, B, ω) is a Cartan geometry modelled on $\mathbf{X} = \partial \mathbf{H}_{\mathbb{K}}^d$, ω is a regular connection, and $Aut(M, \omega)$ does not act properly on M.

Proposition 4. *If (f_k) is a sequence of $Aut(M, \omega)$ which does not act properly on M, then there is an open subset $U \subset M$ and a point $y_\infty \subset M$, such that $\lim_{k \to +\infty} f_k(U) = y_\infty$.*

Proof: Since $Aut(M, \omega)$ does not act properly on M, there is a stable data $(f_k, x_k, x_\infty, y_\infty)$, with (f_k) a sequence of $Aut(M, \omega)$ tending to infinity. Let $(b_k, \hat{x}_k, \hat{x}_\infty, \hat{y}_\infty)$ be an associated holonomy. By theorem 5, the sequence (b_k) tends to infinity in P. So, changing if necessary $(f_k, x_k, x_\infty, y_\infty)$ into the stable data $(f_k^{-1}, y_k, y_\infty, x_\infty)$, Proposition 2 ensures the existence of an open subset $\mathbf{\Omega}^+ \subset [\dot{\mathbf{Q}}]$, such that for any compact subset $K \subset \mathbf{\Omega}^+$, $\lim_{k \to +\infty} b_k.K = [o]$. Let $\mathbf{U}^+ = s(\mathbf{\Omega}^+)$.

As a consequence of Lemma 8 and point (ii) of Proposition 2, we can find $\xi \in \mathbf{U}^+$ such that $\xi_{\hat{x}_\infty} \in \dot{Q}_{\hat{x}_\infty}^{reg}$. In fact, there exists $\epsilon > 0$ small, such that $B_{\xi_{\hat{x}_\infty}}(\epsilon) \subset \dot{Q}_{\hat{x}_\infty}^{reg}$, where $B_{\xi_{\hat{x}_\infty}}(\epsilon) = \omega_{\hat{x}_\infty}^{-1}(B_{\rho_G}(\xi, \epsilon) \cap \mathbf{U}^+)$. This property still

holds if one replaces x_∞ by x_k, for $k \geq k_0$ sufficiently large. In other words, if we put $B_{\hat{x}_k}(\epsilon) = \omega_{\hat{x}_k}^{-1}(B_{\rho_G}(\xi, \epsilon) \cap \mathbf{U}^+)$, then, for $k \geq k_0$, $U_k = Exp_{x_k}(B_{\hat{x}_k}(\epsilon))$ is a sequence of open subsets of M. Moreover, this sequence converges to $U_\infty = Exp_{x_\infty}(B_{\hat{x}_\infty}(\epsilon))$. This ensures the existence of an open subset $U \subset M$ and of $k_1 \geq k_0$, such that $U \subset \bigcap_{k \geq k_1} U_k$.

Now, we suppose ϵ small enough, such that $B_{\rho_G}(\xi, \epsilon) \cap \mathbf{U}^+$ has compact closure in $\mathbf{U}^+$. Let us set $\mathbf{\Lambda}_k = Ad(b_k).(B_{\rho_G}(\xi, \epsilon) \cap \mathbf{U}^+)$. We then have $\lim_{k \to +\infty}[\mathbf{\Lambda}_k] = \lim_{k \to +\infty} b_k.[B_{\rho_G}(\xi, \epsilon) \cap \mathbf{U}^+] = [o]$. We also get, as a consequence of relation (4) in Proposition 3, that $[\Lambda_{\hat{x}_k,k}] = f_k([B_{\hat{x}_k}(\epsilon)])$, . Lemma 7 then gives $\lim_{k \to +\infty} f_k([B_{\hat{x}_k}(\epsilon)]) = [y_\infty]$. Since $Exp_{x_k}(B_{\hat{x}_k}(\epsilon)) \subset [B_{\hat{x}_k}(\epsilon)]$, we get $U \subset [B_{\hat{x}_k}(\epsilon)]$ for all $k \geq k_1$, and the proposition follows.

$\square$

Proposition 5. *Let (f_k)be a sequence of $Aut(M, \omega)$ tending to infinity, and $(f_k, x_k, x_\infty, y_\infty)$ a stable data on M. If the action of (f_k) is equicontinuous at x_∞, then the curvature of ω vanishes at x_∞.*

Proof: Let $(b_k, \hat{x}_k, \hat{x}_\infty, \hat{y}_\infty)$ be an holonomy associated to $(f_k, x_k, x_\infty, y_\infty)$. By Theorem 5, the sequence (b_k) acts equicontinuously at o. By Corollary 1, the attracting and repelling poles o^+ and o^- of (b_k) satisfy $o^+ = o$ and $o^- \neq o$, and we can write $b_k = l_{1,k} a^{t_k} l_{2,k}$, where $l_{1,k}$ and $l_{2,k}$ are two sequences of$P$, relatively compact in P. We will suppose that for $i \in \{1, 2\}$, $l_{i,k}$ converges to $l_{i,\infty} \in P$. Since $o^+ = o$, we have $\lim_{k \to +\infty} t_k = +\infty$.

Let us fix $\eta_1, ..., \eta_n$, a basis of $\mathfrak{n}^-$, and we suppose that $\{1, ..., n\} = I_{-1} \cup I_{-2}$, with $\eta_i \in \mathfrak{n}_1^-$ for$i \in I_{-1}$ and $\eta_i \in \mathfrak{z}^-$ for $i \in I_{-2}$ (of course $I_{-1} = \emptyset$ when $\mathfrak{n}_1^- = \{0\}$). For $i = 1, ..., n$ we set $\xi_{i,k} = Ad(l_{2,k}^{-1}).\eta_i$ (resp. $\zeta_{i,k} = Ad(l_{1,k}).\eta_i$). For every $i = 1, ..., n$, we have $\lim_{k \to +\infty} \xi_{i,k} = \xi_{i,\infty}$ (resp. $\lim_{k \to +\infty} \zeta_{i,k} = \zeta_{i,\infty}$), where $\xi_{i,\infty} = Ad(l_{2,\infty}^{-1}).\eta_i$ (resp. $\zeta_{i,\infty} = Ad(l_{1,\infty}^{-1}).\eta_i$). In the following, we will denote $\mathfrak{h}_k$ (resp. $\mathfrak{h}_\infty$) the vector subspace of $\mathfrak{g}$ spanned by $\xi_{1,k}, ..., \xi_{n,k}$ (resp. $\xi_{1,\infty}, ..., \xi_{n,\infty}$). Since $\mathfrak{h}_k$ is nothing else than the image of $\mathfrak{n}^-$ by $Ad(l_{2,k}^{-1})$, the $\mathfrak{h}_k$'s (resp. $\mathfrak{h}_\infty$) are Lie subalgebras of $\mathfrak{g}$.

Let $(\hat{f}_k)$ be a lift of (f_k) associated to the holonomy (b_k), i.e such that $R_{b_k^{-1}} \circ \hat{f}_k(\hat{x}_k) = \hat{y}_k$. Let us set $\phi_k = R_{b_k^{-1}} \circ \hat{f}_k$. Then, $(\phi_k)^*\Omega = Ad(b_k).\Omega$. We now distinguish two cases.

a) The root system of $\mathfrak{g}$ is $\Delta = \{-\alpha, \alpha\}$. Then, for every $k \in \mathbf{N}$, and $1 \leq i \leq n$, we have $Ad(b_k).\xi_{i,k} = e^{-t_k}\zeta_{i,k}$. We infer the relation $D_{\hat{x}_k}\phi_k(\xi_{\hat{x}_k,i,k}) = e^{-t_k}\zeta_{\hat{y}_k,i,k}$, which implies for every $1 \leq i \leq j \leq n$:

$$Ad(b_k).\Omega_{\hat{x}_k}(\xi_{\hat{x}_k,i,k}, \xi_{\hat{x}_k,j,k}) = e^{-2t_k}\Omega_{\hat{y}_k}(\zeta_{\hat{y}_k,i,k}, \zeta_{\hat{y}_k,j,k})$$

Now, looking at the root space decomposition of $\mathfrak{g}$, there is a $C > 0$ such that for any $u \in \mathfrak{g}$, $||Ad(b_k).u|| \geq e^{-t_k}.||u||$. We infer that $\Omega_{\hat{x}_\infty}(\xi_{\hat{x}_\infty,i,\infty}, \xi_{\hat{x}_\infty,j,\infty})) =$

$\lim_{k\to+\infty}\Omega_{\hat{x}_k}(\xi_{\hat{x}_k,i,k},\xi_{\hat{x}_k,j,k})=0$. The subspace $\mathfrak{h}_\infty$ is a supplementary of $\mathfrak{p}$ in $\mathfrak{g}$, so that $\Omega_{\hat{x}_\infty}=0$.

b) The root system of $\mathfrak{g}$ is $\Delta=\{-2\alpha,-\alpha,\alpha,+2\alpha\}$. Then, $Ad(b_k).\xi_{i,k}=e^{-t_k}\zeta_{i,k}$ for all $i\in I_{-1}$, and $Ad(b_k).\xi_{i,k}=e^{-2t_k}\zeta_{i,k}$ for all $i\in I_{-2}$. By the same calculation as before, we get that:

- $||Ad(b_k).\Omega_{\hat{x}_k}(\xi_{\hat{x}_k,i,k},\xi_{\hat{x}_k,j,k})||\leq e^{-2t_k}\Omega_{\hat{y}_k}(\zeta_{\hat{y}_k,i,k},\zeta_{\hat{y}_k,j,k})$ for all $1\leq i<j\leq n$.
- $||Ad(b_k).\Omega_{\hat{x}_k}(\xi_{\hat{x}_k,i,k},\xi_{\hat{x}_k,j,k})||\leq e^{-3t_k}\Omega_{\hat{y}_k}(\zeta_{\hat{y}_k,i,k},\zeta_{\hat{y}_k,j,k})$ if i or j is in I_{-2}.

Looking at the decomposition of $\mathfrak{g}$ into rootspaces, we see that the converging sequences (u_k) of $\mathfrak{g}$ such that $||Ad(b_k).u_k||$ is contracted at a rate at least e^{-2t_k} must converge in $\mathfrak{h}_\infty$, whereas those sequences (u_k) such that $||Ad(b_k).u_k||$ is contracted at a rate at least e^{-3t_k} must converge to $0_{\mathfrak{g}}$. We infer that $\Omega_{\hat{x}_\infty}(\xi_{\hat{x}_\infty,i,\infty},\xi_{\hat{x}_\infty,j,\infty})=0$ as soon as i or j is in I_{-2}. If both i and j are in I_{-1}, we get that $\Omega_{\hat{x}_\infty}(\xi_{\hat{x}_\infty,i,\infty},\xi_{\hat{x}_\infty,j,\infty})$ is in $l_{2,\infty}^{-1}.\mathfrak{z}^-$. By the hypothesis of regularity on the connection ω, this is possible only if $\Omega_{\hat{x}_\infty}(\xi_{\hat{x}_\infty,i,\infty},\xi_{\hat{x}_\infty,j,\infty})=0$. We finally get that $\Omega_{\hat{x}_\infty}=0$ on $\mathfrak{h}_\infty$, and thus $\Omega_{\hat{x}_\infty}=0$.

□

Let us fix x_∞ a point of the open set U given by Proposition 4. Then, if (x_k) is the sequence constant to x_∞, and $y_k=f_k(x_\infty)$, then $(f_k,x_k,x_\infty,y_\infty)$ is a stable data. We note this peculiar stable data $(f_k,x_\infty,x_\infty,y_\infty)$.

Proposition 6. *The point x_∞ has an open neighbourhood Λ, which is geometrically isomorphic to a quotient $\Gamma\backslash\Omega_o$, with $\Gamma\subset P$ a dicrete subgroup (possibly trivial).*

Proof:

We consider an holonomy data $(b_k,\hat{x}_\infty,\hat{x}_\infty,\hat{y}\infty)$ associated to $(f_k,x_\infty,x_\infty,y_\infty)$. We still call $\phi_k=R_{b_k^{-1}}\circ\hat{f}_k$, and we use the same notations as in Lemma 5. In particular, the action of (f_k) being equicontinuous at x_∞, b_k writes $l_{1,k}a^{t_k}l_{2,k}$ with $\lim_{k\to+\infty}t_k=+\infty$.

For every $r\in\mathbb{R}_+^*$, and $k\in\mathbf{N}\cup\{\infty\}$, we denote by $B_{r,k}$ the intersection of $\mathfrak{h}_k$ with the ball of center $0_{\mathfrak{g}}$ and radius r (for the norm $||.||$). For every $p\in B$, we set $H_{p,\mathfrak{h}_k}=\omega_p^{-1}(\mathfrak{h}_k)$, and $B_{p,r,k}=\omega_p^{-1}(B_{r,k})$. Since $\hat{x}_k$ and $\hat{y}_k$ are both converging sequences, one can find $r\in\mathbb{R}_+^*$ such that $B_{r,k}\subset\mathbf{W}_{\hat{x}_k}\cap\mathbf{W}_{\hat{y}_k}$ for every $k\in\mathbf{N}\cup\{\infty\}$.

Lemma 11. *There is a sequence of integers $(k_m)_{m\in\mathbf{N}}$ such that :*

- *for every $m\in\mathbf{N}$, $B_{\hat{x}_\infty,m,k_m}\subset W^B_{\hat{x}_\infty}$, and $Exp_{\hat{x}_\infty}$ maps $B_{\hat{x}_\infty,m,k_m}$ diffeomorphically on its image (this image is denoted by U_m).*
- *$V_m=[B_{m,k_m}]$ is a strictly increasing sequence of open subsets of $\mathbf{X}\setminus\{o^-\}$, the union of which is $\mathbf{X}\setminus\{o^-\}$*

Proof:

From the relation $\phi_k \circ Exp_{\hat{x}_\infty} = Exp_{\hat{y}_k} \circ D_{\hat{x}_\infty}\phi_k$, we infer that $D_{\hat{x}_\infty}\phi_k$ maps $W^B_{\hat{x}_\infty}$ on $W^B_{\hat{y}_k}$. Lets us fix m. Since $Ad(b_k)(B_{m,k})$ tends to $0_{\mathfrak{g}}$ as $k \to +\infty$, $D_{\hat{x}_\infty}\phi_k(B_{\hat{x}_\infty,m,k})$ tends to $0_{\hat{y}_\infty}$ as $k \to +\infty$. Thus, there exists an integer $k_{1,m} \in \mathbf{N}$ such that $Exp_{\hat{y}_k}$ maps $D_{\hat{x}_\infty}\phi_k(B_{\hat{x}_\infty,m,k})$ diffeomorphically on its image, when $k \geq k_{1,m}$. We infer that $Exp_{\hat{x}_\infty}$ maps $B_{\hat{x}_\infty,m,k}$ diffeomorphically on its image. On the other hand, as $k \to +\infty$, $[B_{m,k}]$ tends to $[B_{m,\infty}]$, which has compact closure in $\pi_X \circ Exp_G(Ad(\mathfrak{h}_\infty) = \mathbf{X} \setminus \{o^-\}$. So, taking $k \geq k_{2,m}$, we can suppose that $[B_{m,k}] \subset \mathbf{X} \setminus \{o^-\}$. Finally, since $[B_{m,\infty}]$ is a strictly increasing sequence, one can find a sequence (k_m), with $k_m \geq \max(k_{1,m}, k_{2,m})$ for every m, such that $[B_{m,k_m}]$ is also a strictly increasing sequence, for the inclusion. □

The "North-South" dynamics properties for the sequence (b_k) imply that $\lim_{k\to+\infty} b_k.[B_{m,k_m}] = [o]$. Lemma 7 then gives: $\lim_{k\to+\infty} f_k([B_{\hat{x}_\infty,m,k_m}]) = [y_\infty]$. We conclude that the action of (f_k) is equicontinuous at each point $x \in U_m$. As a consequence of Proposition 5, the curvature vanishes on U_m.

We are now in the following situation. The open set $U_\infty = \bigcup_{m=1}^{\infty} U_m$ is flat and contains x_∞. Let $\tilde{U}_\infty$ be its universal cover. Let $\tilde{x}_\infty \in \tilde{U}_\infty$ be a point over x_∞, and $\delta : \tilde{U}_\infty \to \mathbf{X}$ a developping map, mapping $\tilde{x}_\infty$ on o (see section 3.3.1). For every $m \in \mathbf{N}$, U_m can be lifted to an open subset $\tilde{U}_m \subset \tilde{U}_\infty$ in the following way: for every $\zeta \in B_{\hat{x}_\infty,m,k_m}$, the curve ζ^* has a unique lift $\tilde{\zeta}^* \in C^1(\tilde{U}_\infty, [0,1], \tilde{x}_\infty)$. The open set $\tilde{U}_m$ is the union of such lifts. Since for every $\zeta \neq \xi$ in $B_{\hat{x}_\infty,m,k_m}$, $\zeta^*(1) \neq \xi^*(1)$ (indeed Exp_{x_∞} maps $B_{\hat{x}_\infty,m,k_m}$ diffeomorphically on its image), the projection of $\tilde{U}_m$ on U_m is a diffeomorphism.

We use the results of section section 3.3.1. Let $\hat{x}'_\infty = R_{b_0}.\hat{x}_\infty$ such that $\tilde{\delta}(\hat{x}'_\infty) = e$. Then $\delta(\tilde{\zeta}^*) = \mathcal{D}^{\hat{x}'_\infty}_{x_\infty}(\zeta^*) = b_0.\mathcal{D}^{\hat{x}_\infty}_{x_\infty}(\zeta^*)$, so that $\delta(\tilde{U}_m) = b_0.V_m$.

Lemma 12. *The map δ is injective on $\tilde{U}_\infty$, and realizes a geometrical isomorphism between $\tilde{U}_\infty$ and $\mathbf{X} \setminus \{b_0.o^-\}$.*

Proof: We first prove that for all $m \in \mathbf{N}$, δ is injective on $\tilde{U}_m$. Let $\tilde{x}$ and $\tilde{y}$ be distinct points in $\tilde{U}_m$. They project on two distinct points x and y in U_m. We write $x = Exp_{x_\infty}(\zeta_{\hat{x}_\infty,m,k_m})$ and $y = Exp_{x_\infty}(\xi_{\hat{x}_\infty,m,k_m})$ with $\zeta \neq \xi$ in B_{m,k_m}. Thus, $\delta(x) = \zeta^*(1)$ and $\delta(y) = \xi^*(1)$. But $\pi_X \circ Exp_G$ is a diffeomorphism from B_{m,k_m} on its image. As a consequence, $\delta(x) \neq \delta(y)$.

To conclude, we just remark that $\tilde{U}_m$ is an increasing sequence of open subsets. □

It follows from the previous lemma that $\tilde{U}_\infty$ is geometrically isomorphic to the open set $\mathbf{X} \setminus \{b_0.o^-\}$, which is itself geometrically isomorphic to Ω_o.

We get that U_∞ is geometrically isomorphic to a quotient $\Gamma\backslash\Omega_o$, where $\Gamma \subset P$ is a discrete subgroup acting freely properly discontinuously on Ω_o.

$\square$

We conclude the proof of Theorem 3 thanks to Lemma 10, and the following result, applied to the inclusion $U_\infty \subset M$:

Theorem 6. *Let (M, B, ω) be a Cartan geometry modelled on $\mathbf{X} = \partial\mathbf{H}^d_{\mathbb{K}}$. Let Γ be a discrete subgroup of P, acting freely properly discontinuously on Ω_o. We suppose that there is a geometric embedding σ from $\Gamma\backslash\Omega_o$ into M. Then we are in one of the following cases :*

(i) M is geometrically isomorphic to the model space $\mathbf{X}$. In this case, $\Gamma = \{e\}$, and there is a point $\kappa \in M$ such that $\sigma(\Omega_o) = M \setminus \{\kappa\}$.

(ii) The embedding σ is a geometrical isomorphism between $\Gamma\backslash\Omega_o$ and M.

The proof of this result will be the aim of the last section of this article. As an illustration, let us remark that Theorem 6, when applied to the case of conformal riemannian structures, yields the:

Corollary 3. *Let us suppose that σ is a conformal embedding from a flat, complete, Riemannian manifold M of dimension $n \geq 3$, into a Riemannian manifold N of the same dimension. Then, either σ is a conformal diffeomorphism between M and N, or M is the Euclidean space E_n, and N is conformally diffeomorphic to the standard sphere $\mathbf{S}^n$. In this case, σ is the composition of the standard stereographic projection, with some Möbius transformation.*

7 A rigidity theorem for the geometric embeddings

This last part is devoted to the proof of Theorem 6. Other cases of rigidity for geometrical embeddings of certain Cartan geometries will be studied more extensively in [Fr1].

7.1 Geometrical embeddings

Let (M, B, ω) and (N, B', ω') be two manifolds endowed with Cartan geometries modelled on $\mathbf{X} = G/P$ (we are speaking here of the most general framework : G is any Lie group and P is a closed subgroup of G). In particular, M and N have the same dimension.

A map σ will be called *a geometrical embedding* from M to N if σ lifts into a fiber bundle embedding $\hat{\sigma} : B \to B'$, satisfying $\hat{\sigma}^*\omega' = \omega$.

7.2 Cauchy boundary of a Cartan geometry

We decribe here a way for attaching a boundary to any Cartan geometry (M, B, ω), modelled on a space $\mathbf{X} = G/P$. This method was already used in general relativity, to associate a boundary to spacetimes (see for example [S]).

We already saw in section 3 that the choice of a basis of $\mathfrak{g}$ yielded a parallelism $\mathcal{R}$, and a Riemannian metric ρ on B. The parallelism is orthonormal for the metric ρ. Let us denote by d_ρ the distance associated to the metric ρ. We can consider $(\overline{B}, \overline{d})$, *the Cauchy completion* of the metric space (B, d_ρ). For every $b \in P$, the differential of R_b, when expressed in the trivialisation of TB given by the parallelism $\mathcal{R}$, is just the linear transformation $Ad(b^{-1})$. As a consequence, R_b is a uniformly continuous transformation of (B, d_ρ) for all $b \in P$. Thus, it can be prolongated into a continuous transformation $\overline{R}_b$ of $(\overline{B}, \overline{d})$. The space $\overline{B}$ writes naturally as the union $\overline{B} = B \cup \partial_c B$. The *Cauchy boundary* $\partial_c B$ is stable for the action of the $\overline{R}_b$'s, $b \in P$. We then set $\partial_c M = \partial_c B/P$ (the quotient is taken for the action of P on $\partial_c B$ by the transformations $\overline{R}_b$).

Of course, in general, the space $M \cup \partial_c M = \overline{B}/P$ behaves very badly from a topological point of view (it is generally non Hausdorff), so that the previous construction has a limited interest to get "nice boundaries".

Let us illustrate this construction in the case $\mathbf{X} = \partial \mathbf{H}^d_{\mathbb{K}}$.

Lemma 13.
(i) $\partial_c \mathbf{X} = \emptyset$.
(ii) If $\Gamma \subset G$ *is a discrete subgroup acting freely properly discontinuously on* Ω_o*, the boundary* $\partial_c(\Gamma \backslash \Omega_o)$ *reduces to a point.*

Proof: Let us recall that for the model space $\mathbf{X}$, the Cartan bundle is just the group G, and the choice of a basis of $\mathfrak{g}$ determines a unique left invariant Riemannian metric ρ_G (making the given basis an orthonormal basis). We denote by B_o the preimage of Ω_o by the fibration $\pi_X : G \to \mathbf{X}$. The topological boundary ∂B_o of this open subset in G is just the fiber over o. Let ρ_o be the metric induced by ρ_G on B_o.

The Riemannian manifold (G, ρ_G) is homogeneous, hence complete, what proves point (i). The group Γ acts by isometries for ρ_G. The manifold $\Gamma \backslash G$ is endowed with a metric $\overline{\rho}_G$, induced by ρ_G via the covering map. The manifold $(\Gamma \backslash G, \overline{\rho}_G)$ is also complete. The quotient $\overline{B}_o = \Gamma \backslash B_o$ can be identified with an open subset of $\Gamma \backslash G$, and its topological boundary $\partial \overline{B}_o \subset \Gamma \backslash G$ is a submanifold $\Gamma \backslash \partial B_o$. Since $\partial \overline{B}_o$ is a closed submanifold of $\Gamma \backslash G$, any point $p_\infty \in \partial \overline{B}_o$ is obtained as the limit of a Cauchy sequence (p_n) of $\overline{B}_o$ (for $d_{\overline{\rho}_o}$, the metric induced by ρ_o via the covering map). Reciprocally,

every sequence of $\overline{B}_o$, which is a Cauchy sequence for $d_{\overline{\rho}_o}$, is also a Cauchy sequence for d_{ρ_G}. Thus, $(\Gamma\backslash G, d_{\overline{\rho}_G})$ is the Cauchy completion of $(\overline{B}_o, d_{\overline{\rho}_o})$, and $\partial_c\overline{B}_o$ can be identified with $\partial\overline{B}_o$. Since P acts transitively on $\partial\overline{B}_o$, we get that $\partial_c(\Gamma\backslash\Omega_o)$ is just one point.

$\square$

7.3 The geometrical boundary of an embedding

Let (M, B, ω) and (N, B', ω') be two Cartan geometries modelled on the same space $\mathbf{X} = G/P$.

We assume that there is a geometrical embedding $\sigma : M \to N$. Then, the topological boundary of $\sigma(M)$ in N is called the *geometrical boundary of M, associated with the embedding σ*, and denoted $\partial_\sigma M$. We will also write $\partial_{\hat{\sigma}} B$ for the topological boundary of $\hat{\sigma}(B)$ in B'.

We call ρ and ρ' the Riemanniann metrics on B and B', associated *to the choice of a same basis of* $\mathfrak{g}$, as they were defined in section 3.1. For these metrics, the embedding $\hat{\sigma}$ is in fact an isometric embedding: $\hat{\sigma}^*\rho' = \rho$. We call d_ρ and $d_{\rho'}$ the distances associated to ρ and ρ'.

Definition 12. *We call $d^\sigma_{\rho'}$ the distance on $\hat{\sigma}(B)$ defined by:*

$$d^\sigma_{\rho'}(p, q) = \inf\{L_{\rho'}(\gamma) \mid \gamma \in C^1([0,1], \hat{\sigma}(B)),\ \gamma(0) = p,\ \gamma(1) = q\}$$

Let us remark that $\hat{\sigma}$ is an isometry from (B, d_ρ) to $(\hat{\sigma}(B), d^\sigma_{\rho'})$. In particular, by equivariance of the action of P, the transformations R_b, for $b \in P$, are uniformously continuous for the distance $d^\sigma_{\rho'}$.

We will need also the following:

Definition 13 (Regular points). *We say that a point $p \in \partial_{\hat{\sigma}} B$ is regular if there exist a sequence (p_n) of $\hat{\sigma}(B)$ which tends to p, and such that (p_n) is a Cauchy sequence for the distance $d^\sigma_{\rho'}$.*

- A point $x \in \partial_\sigma M$ is said to be regular if there exist a regular point $\partial_{\hat{\sigma}} B$ over x.

- The set of regular ponts of $\partial_{\hat{\sigma}} B$ (resp. $\partial_\sigma M$) is denoted by $\partial^{reg}_{\hat{\sigma}} B$ (resp. $\partial^{reg}_\sigma M$).

Let us remark that since $\partial^{reg}_{\hat{\sigma}} B$ is invariant under the action of P on B', if there exist a point of $\partial^{reg}_{\hat{\sigma}} B$ over x, then all the points of the fiber over x are regular.

Lemma 14. *If $\partial_\sigma M$ is not empty, then $\partial^{reg}_\sigma M$ is dense in $\partial_\sigma M$.*

Proof: Let us pick a point $x_\infty \in \partial_\sigma M$, and lift it into a point $\hat{x}_\infty \in \partial_{\hat{\sigma}} B$. We fix a small ball $B_{\rho'}(0_{\hat{x}_\infty}, \epsilon) \subset T_{\hat{x}_\infty} B'$ such that $Exp_{\hat{x}_\infty}$ maps $B_{\rho'}(0_{\hat{x}_\infty}, \epsilon)$ diffeomorphically on its image. Then, there exists $u_0 \in B_{\rho'}(0_{\hat{x}_\infty}, \epsilon)$ such that $\hat{u}_0^*(]0,1]) \cap \hat{\sigma}(B) \neq \emptyset$. Indeed, if it were not the case, the open set $Exp_{x_\infty}(B_{\rho'}(0_{\hat{x}_\infty}, \epsilon))$ would not intersect $\sigma(M)$, contradicting $x_\infty \in \partial_\sigma M$.

Now, if there is a $t_0 \in]0,1]$ such that $\hat{u}_0^*(]0,t_0]) \subset \hat{\sigma}(B)$, it means that $\hat{x}_\infty$ itself is regular. Indeed, if (t_n) is a sequence of $]0,t_0]$ tending to 0, the sequence $\hat{u}_0^*(t_n)$ is a Cauchy sequence for $d^\sigma_{\rho'}$ and tends to $\hat{x}_\infty$.

If a t_0 as above does not exist, then $\hat{u}_0^*(]0,1]) \cap \hat{\sigma}(B)$ has infinitely many connected components. There is a decreasing sequence of $]0,1]$, let us call it $(t_n)_{n\in\mathbf{N}}$, converging to 0, and such that those connected components are the intervals $\hat{u}_0^*(]t_{2k+1}, t_{2k}[)$, $k \geq 0$. By the same argument as above, the points $\hat{u}_0^*(t_n)$ are in $\partial_{\hat{\sigma}}^{reg} B$, for $n \geq 1$. We get a sequence of regular points tending to $\hat{x}_\infty$. Projecting on N, we get a sequence of $\partial_\sigma^{reg} M$ converging to x_∞.

$\square$

Lemma 15. *There is a map $\hat{\jmath} : \partial_{\hat{\sigma}}^{reg} B \to \mathcal{P}(\partial_c B)$ (where $\mathcal{P}(\partial_c B)$ stands for the set of parts of $\partial_c B$), such that if $\hat{x}$ and $\hat{y}$ are two distinct points of $\partial_\sigma^{reg} B$, then $\hat{\jmath}(x) \cap \hat{\jmath}(y) = \emptyset$.*

Moreover, the map $\hat{\jmath}$ is equivariant for the action of P on $\partial_{\hat{\sigma}}^{reg} B$ and $\mathcal{P}(\partial_c B)$. It thus induces an injective map $j : \partial_\sigma^{reg} M \to \mathcal{P}(\partial_c M)$.

Proof: We define the map $\hat{\jmath}$ in the following way: for every point $\hat{x}_\infty \in \partial_{\hat{\sigma}}^{reg} B$, $\hat{\jmath}(\hat{x}_\infty)$ is the set of sequences $(\hat{x}_k)$ of $\hat{\sigma}(B)$, converging to $\hat{x}_\infty$, and which are Cauchy sequences for $d^\sigma_{\rho'}$ (to be precise, the points of $\partial_c B$ are rather defined by the sequences $\hat{\sigma}^{-1}(\hat{x}_k)$). With this definition, if $b \in P$, it is clear that $\hat{\jmath}(R_b.\hat{x}_\infty)$ is the part $R_b.\hat{\jmath}(\hat{x}_\infty)$, showing the equivariance of $\hat{\jmath}$. If $\hat{x}_\infty$ and $\hat{y}_\infty$ are distinct in $\partial_{\hat{\sigma}}^{reg} B$, and if $(\hat{x}_k)$ and $(\hat{y}_k)$ are two sequences of $\hat{\jmath}(\hat{x}_\infty)$ and $\hat{\jmath}(\hat{y}_\infty)$ respectively, then there is $\epsilon > 0$ such that for $k \in \mathbf{N}$, $d_{\rho'}(\hat{x}_k, \hat{y}_k) > \epsilon$. Hence, *a fortiori*, $d^\sigma_{\rho'}(\hat{x}_k, \hat{y}_k) > \epsilon$, what proves that $(\hat{x}_k) \neq (\hat{y}_k)$ in $\partial_c B$.

Let us now check that the induced map j is also injective. Let us pick $x \neq y$ in $\partial_\sigma^{reg} M$, and choose $\hat{x}$ and $\hat{y}$ over x and y respectively. If $j(x) \cap j(y) \neq \emptyset$, it means that $\hat{\jmath}(\hat{x}) \cap R_b(\hat{\jmath}(\hat{y})) \neq \emptyset$, for some $b \in P$. But by equivariance, it means that $\hat{\jmath}(\hat{x}) \cap \hat{\jmath}(R_b.\hat{y}) \neq \emptyset$, a contradiction since $\hat{x} \neq R_b.\hat{y}$.

$\square$

7.4 Proof of Theorem 6

We assume that the hypothesis of theorem 6 are satisfied, and we call M the quotient $\Gamma\backslash\Omega_o$.

The first case to deal with is $\partial_\sigma M = \emptyset$. In this case, σ is a diffeomorphism. It is thus a geometric isomorphism between M and N, and we are in the case (ii) of the theorem.

Now, let us assume $\partial_\sigma M \neq \emptyset$. By Lemma 14, the set of regular points of $\partial_\sigma M$ is dense in $\partial_\sigma M$, and in particular, is nonempty. But Lemma 15, together with point (ii) of Lemma 13, ensures that the set of regular points is a singleton, and moreover, has to be dense in $\partial_\sigma M$. This proves that $\partial_\sigma M$ itself has just one point, that we call κ. Thus, $\partial_{\hat{\sigma}}^{reg} B = \partial_{\hat{\sigma}} B$ can be identified with the fiber (i.e a P-orbit) of B' over κ. The map $\hat{\jmath}$ of Lemma 15 is an equivariant map from this fiber, onto $\partial_c B = \Gamma \backslash P$. This forces the action of P on $\Gamma \backslash P$ to be free, which implies $\Gamma = \{e\}$.

Finally, $N = \sigma(\Omega_o) \cup \{\kappa\}$. So, N is diffeomorphic to $\mathbf{X}$, hence simply connected. Moreover, (M, B, ω) is flat since it is flat on a dense open set. Thus, there is a developping map from N to $\mathbf{X}$, which turns out to be a covering map, by compacity of N. This developping map is then a geometrical isomorphism between N and $\mathbf{X}$.

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